A tour d'horizon of de Casteljau's (geometrical) work

Visual Computing Colloquium
Otto-von-Guericke-Universität Magdeburg

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Paul Bertrand René Lucien de Faget de Casteljau (1930-2022)

The course of his life


Comte A F Fourcroy
1755-1809


A Floucaud de Fourcroy
1831-1929

1930	Born on November 19 in Besançon, *Ecole maternelle, Collège*
1940	Fled from the Germans to Agen, *Collège*
1945	Returned to Besançon, *College, Lycée, Classes préparatoires (Maths sup, spé)*
1951-55	Student at *Ecole Normale Supérieure* in Paris (worldwide highest Nobel prize density per student)
1955-58	Military service in France and Algeria
1958	Joined Citroën, working in the department for Calculation Methods in Tooling (*)
1963	Married Aliette Dubosc de Pesquidoux
1974	Demotion following Peugeot's takeover of Citroën
1992	Retired
1991+	Invited to give lectures (1991 Biri, 1992 Oberwolfach, 1993 Chamonix, 1994 Ulvik, 1996 Dagstuhl, 1996 Chamonix, 1999 St Malo, 2000 MICAD, 2001 MICAD)
1997	Awarded honorary doctorate (Dr. phil. h.c.) by the University of Bern; further awards: 1987 Seymour Cray Prize, 1996 John Gregory Award, 1998 SIGGRAPH Pioneer, 2012 Bézier Award
2022	Died on March 24 in Versailles, only a few weeks after his wife

* Why Citroën? “Let's say, I went to the most urgent one, the company that was hiring as quickly as possible and which was, I'll add, closest to the room I had found. There you go.” (Poitou 1968)

His (mostly French) Publications:

4 **monographies**
1985. Formes à Pôles. Mathématiques et CAO, vol.2. Hermès, Paris.
1986. **Shape Mathematics and CAD**. Mathematics and CAD, vol.2. Kogan Page, London.
1987. Les Quaternions. Traité des Nouvelles Technologies. Série Mathématiques appliquées. Hermès, Paris.
1990. Le Lissage. Traité des Nouvelles Technologies. Série Mathématiques appliquées. Hermès, Paris.

12 **articles**
1956, mit J. Friedel: Étude de la résistivité et du pouvoir thermoélectrique des impuretés dissoutes dans les métaux nobles. J. Phys. Radium 17, 27–32.
1992a. **POLynomials, POLar Forms, and InterPOLation**. In: Lyche, T., Schumaker, L.L. (Eds.), Mathematical Methods in Computer Aided Geometric Design II. Academic Press Professional Inc., pp.57–68. Translated by Hans-Peter Seidel.
1993. **Polar forms for curve and surface modeling as used at Citroën**. In: Piegl, L. (Ed.), Fundamental Developments of Computer-Aided Geometric Modeling. Academic Press, pp .1–12.
1994. Splines Focales. In: Laurent, P., le Méhauté, A., Schumaker, L. (Eds.), Curves and Surfaces in Computer Aided Geometric Design II. AK Peters, pp.91–103.
1995a. Courbes et Profils Esthétiques contre Fonctions Orthogonales (Histoire Vécue). In: Dæhlen, M., Lyche, T., Schumaker, L.L. (Eds.), Mathematical Methods for Curves and Surfaces. Vanderbilt University Press, pp.73–82.
1997. La Tolérance d'Usinage chez Citroën dans les Années (19)60. In: le Méhauté, A., Schumaker, L.L., Rabut, C. (Eds.), Curves and Surfaces with Applications in CAGD. Vanderbilt University Press, pp.69–76.
1998. **Intersection methods of convergence**. In: Farin, G., Bieri, H., Brunnett, G., De Rose, T. (Eds.), Geometric Modelling. Springer Vienna, Vienna, pp.77–80.
1999a. **De Casteljau's autobiography: My time at Citroën**. Computer Aided Geometric Design 16, 583–586. Translation based on Müller and Schultze (1995).
1999b. In mémoriam Henri de Faget de Casteljau: Son autre passe-temps, la géométrie à travers l'hexagone de Pascal. Procès-verbaux et Mémoires de l'Académie des Sciences, Belles Lettres et Arts de Besançon et de Franche-Comté 193, 91–114.
2000. Intersections et Convergence. In: Laurent, P., Sablonnière, P., Schumaker, L.L. (Eds.), Curve and Surface design: Saint-Malo 99. Vanderbilt University Press, pp.9–15.
2001a. Au-delà du Nombre d'OR. Revue internationale de CFAO et d'informatique graphique 16, 19–31.
2001b. Fantastique strophoïde rectangle. Revue internationale de CFAO et d'informatique graphique 16, 357–370.

3 **internal reports**
1959. Améliorations Possibles Apportées aux Techniques de Cotation et de Calcul Numérique (= Enveloppe Soleau 40.040.) Document no P 2108, registered at INPI Paris. 12 pages. S.A. André Citroën.
1963. Courbes et Surfaces à Pôles. Document no P.4147, registered with the bailiff, unpublished., 58 pages. S.A. André Citroën.
1966. Calcul de la Tolérance. Unpublished manuscript of 18 pages. S.A. André Citroën.

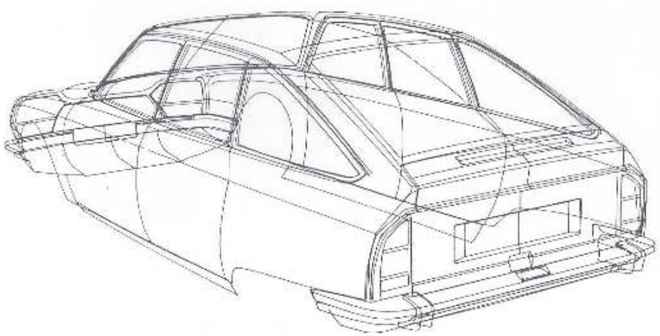
4 **manuscripts**
1983. Formes à Pôles: Courbes et Surfaces à Pôles. Unpublished manuscript of 34 pages. Probably submitted to the Académie des Sciences.
1992b. Mon temps chez Citroën. Manuscript of 3 pages on the occasion of his retirement. Translated into German (Müller and Schultze, 1995) and English (de Faget de Casteljau, 1999a).
1995b. Géométrie Métrique: Essai de définition. Unpublished fragmented manuscript of 80 pages.
1997 Le Récit de mon aventure. 27 pages. published by Müller (2024a) **The story of my adventure** Computer Aided Geometric Design 110, 102278

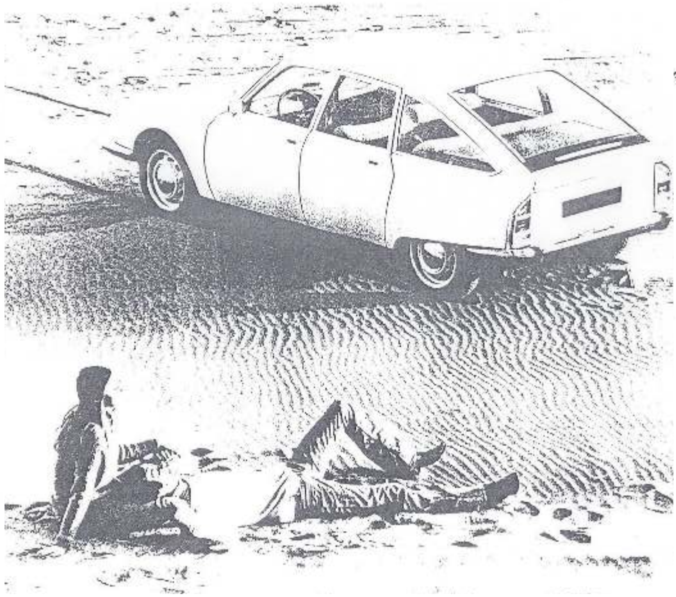
as well as numerous letters

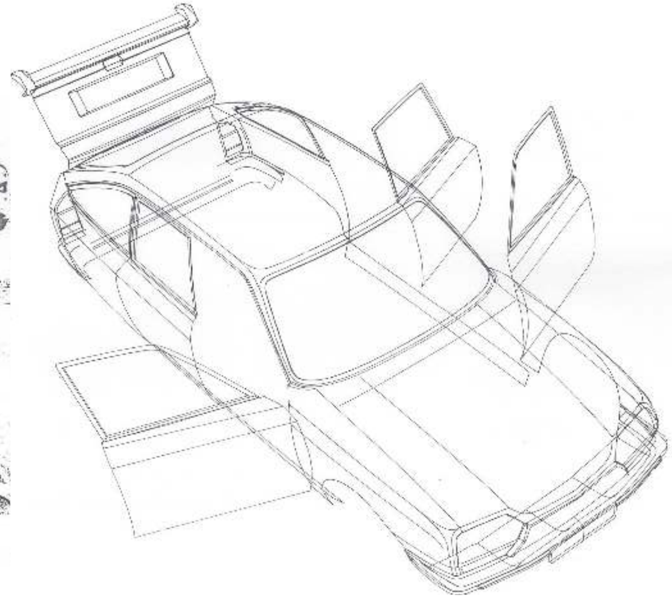
English texts are highlighted in red.

Citroën GS

The first car with a completely CAD-designed body (August 1968).*







* The body of Peugeot 104 was finished six months later, end 1968, early 1969, based on UNISURF.

(Poitou 1989:39)

His (main) Geometric Works

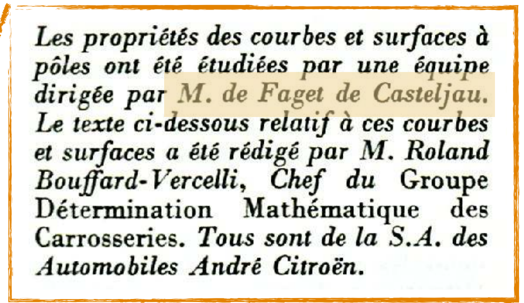
“The” algorithm
Algebraic smoothing
Metric geometry:
    focal splines, 14-point strophoid, conjugate mirrors, regular polygons
16D+3C

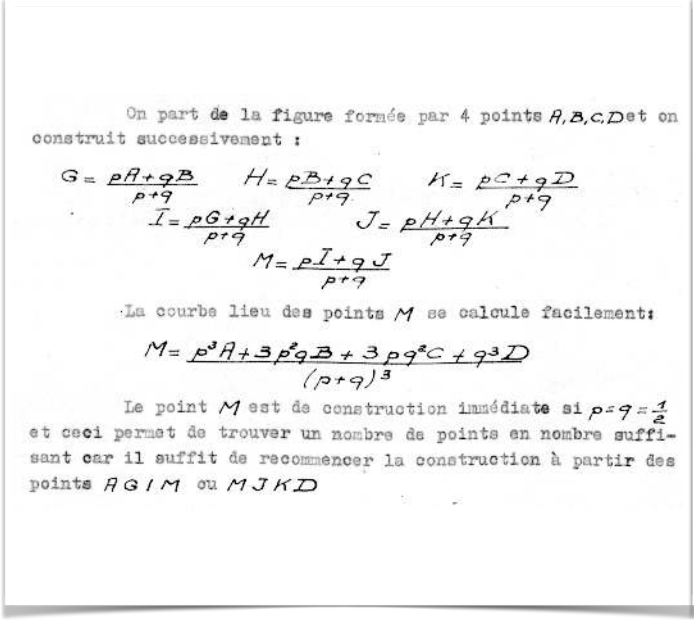
De Casteljau's algorithm

"The" algorithm
Algebraic smoothing
Metric geometry:
- focal splines, 14-point strophoid, conjugate mirrors, regular polygons

16D+3C

"The" Algorithm (1959)





On part de la figure formée par 4 points A,B,C,D et on construit successivement :

$$G=\frac{pA+qB}{p+q} \quad H=\frac{pB+qC}{p+q} \quad K=\frac{pC+qD}{p+q}$$

$$I=\frac{pG+qH}{p+q} \quad J=\frac{pH+qK}{p+q}$$

$$M=\frac{pI+qJ}{p+q}$$

La courbe lieu des points M se calcule facilement:

$$M=\frac{p^3A+3p^2qB+3pq^2C+q^3D}{(p+q)^3}$$

Le point M est de construction immédiate si $p=q=\frac{1}{2}$ et ceci permet de trouver un nombre de points en nombre suffisant car il suffit de recommencer la construction à partir des points $AGIM$ ou $MJKD$

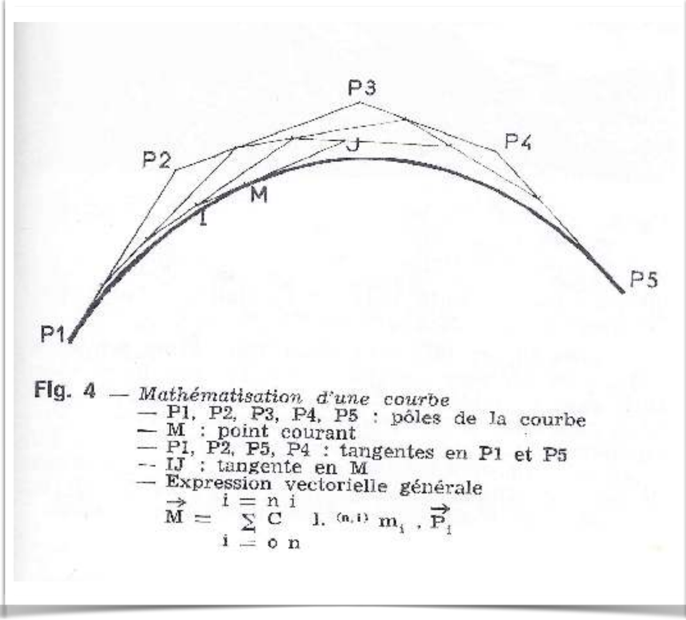
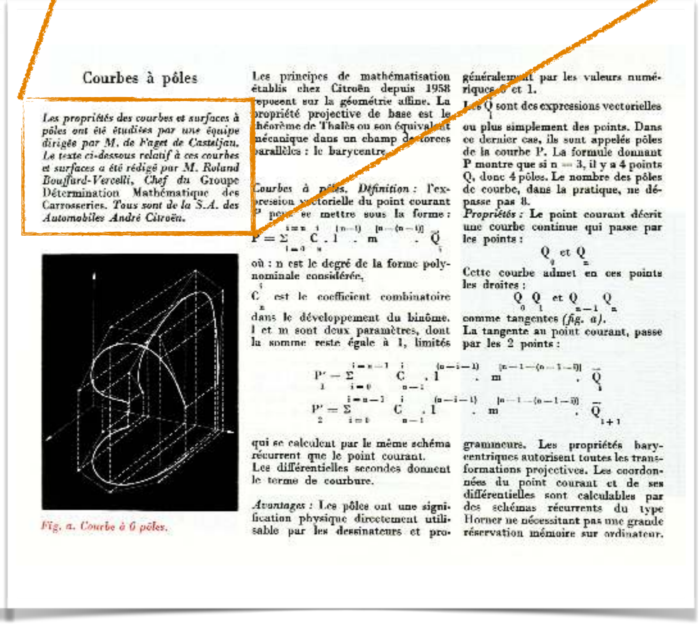


Fig. 4 — *Mathématisation d'une courbe*
— P1, P2, P3, P4, P5 : pôles de la courbe
— M : point courant
— P1, P2, P5, P4 : tangentes en P1 et P5
-- IJ : tangente en M
— Expression vectorielle générale

$$\vec{M} = \sum_{i=0}^{i=n} C \; 1.^{(n,i)} \, m_i \, . \, \vec{P}_i$$



1959, internal report

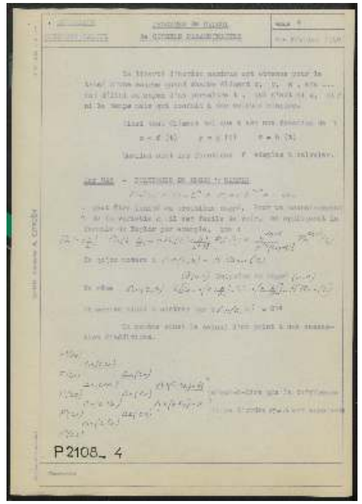
1971, "Krautter/Parizot"
(1975 discovered by Boehm)

1971-75 (encyclopedia),
Bouffard-Vercelli mentions
de Faget de Casteljau

Bézier: 1966, 1967, 1968abcd, 1969, 1970, 1971, 1972ab, 1973, 1974, 1976, 1977 (Diss.), ...

Iterative Functions I (already 1959)

Evaluate a given polynomial (extrapolation)


P 2108_ 4

Polynomial: $p_n(t) = a_n t^n + a_{n-1} t^{n-1} + \ldots a_0$

using central differences and Taylor:

$$p_n(t+\frac{h}{2}) - p_n(t-\frac{h}{2}) = hp_n'(t) + \frac{h^3}{2^2 3!}p_n'''(t) + \ldots \frac{h^{2k+1}}{2^{2k}(2k+1)!}p_n^{2k+1}(t)$$

as well as the recursions

$$q(t) = q(t,h) = p_n(t+\frac{h}{2}) - p_n(t-\frac{h}{2}) = hq_{n-1}(t)$$

or

$$r(t) = q(t+\frac{h}{2}) - q(t-\frac{h}{2}) = h\left[q_{n-1}(t+\frac{h}{2}) - q_{n-1}(t-\frac{h}{2})\right] = h^2 r_{n-2}(t)$$

Thus, the values of a cubic polynomial $P(t)$ are calculated by an initialisation (calculate $n+1=4$ points as well as $\frac{n(n+1)}{2} = 6$ differences) and by $n=3$ additions for each step h:

$$\begin{array}{rcl} p(t+h) & = & p(t) + q(t+\frac{1}{2}h) \\ q(t+\frac{3}{2}h) & = & q(t+\frac{1}{2}h) + r(t+h) \\ r(t+2h) & = & r(t+h) + K \\ p(t+2h) & = & p(t+h) + q(t+\frac{3}{2}h) \\ \text{etc} & & \ldots \end{array}$$

Triangular Scheme:

$$\left(P_i := p(t+i\,h), Q_{i/2} := q(t+\frac{i}{2}h), \cdots\right)$$

$$\begin{array}{llll} P_0 & & & \\ & Q_{1/2} = P_1 - P_0 & & \\ P_1 & & R_1 = Q_{3/2} - Q_{1/2} & \\ & Q_{3/2} = P_2 - P_1 & & S_{3/2} = R_2 - R_1 = K \\ P_2 & & R_2 = Q_{5/2} - Q_{3/2} & \\ & Q_{5/2} = P_3 - P_2 & & S_{5/2} = K \\ P_3 & & R_3 = R_2 + K & \\ & Q_{7/2} = Q_{5/2} + R_3 & & \\ P_4 = P_3 + Q_{7/2} & & R_4 = R_3 + K & \end{array}$$

(de Casteljau 1959, Rabut 2024)

Iterative Functions II (already 1959)

Evaluate a given circular (trig) function I

Circular function: $f(t) = a\cos t + b\sin t$

with angular sum (/diff.) identities:

$$\begin{aligned} \sin(t+h) + \sin(t-h) &= 2\sin t \cos h \\ \cos(t+h) + \cos(t-h) &= 2\cos t \cos h \end{aligned} \quad (*)$$

as well as recursions

$$\begin{aligned} e(t,h) &= f(t+h/2) - f(t-h/2) \\ d(t,h) &= e(t+h/2,h) - e(t-h/2,h) &&= f(t+h) - 2f(t) + f(t-h) \\ &\overset{(*)}{=} 2\cos h(a\cos t + b\sin t) - 2f(t) &&= -2(1-\cos h)f(t) = -Kf(t) \end{aligned}$$

it follows $f(t+h) = 2\cos h f(t) - f(t-h)$ or $f_{n+2} = Kf_{n+1} - f_n$

Thus the values of a circular function $f(t)$ are calculated by an initialisation and by 2 additions (and 1 multiplication by K) for each step h:

$$\begin{aligned} f(t+h) &= f(t) + e(t+\tfrac{1}{2}h) \\ e(t+\tfrac{3}{2}h) &= e(t+\tfrac{1}{2}h) - Kf(t+h) \\ f(t+2h) &= f(t+h) + e(t+\tfrac{3}{2}h) \\ \text{etc} & \quad \dots \end{aligned}$$

Triangular Scheme

$\left(F_i := f(t+ih), E_{i/2} := e(t+\frac{i}{2}h), \cdots\right)$

$$\begin{array}{ccc} F_0 & & \\ & E_{1/2} = F_1 - F_0 & \\ F_1 & & D_1 = -KF_1 \\ & E_{3/2} = E_{1/2} + D_1 & \\ F_2 = F_1 + E_{3/2} & & D_2 = -KF_2 \end{array}$$

(de Casteljau 1959)

Iterative Functions III (1959)

Evaluate a given circular (trig) function II (sin, cos)

Remark: The CORDIC algorithm by Volder (Hewlett-Packard) was published only end of 1959:

circular function: $f(t) = a\cos t + b\sin t$

with angular sum (/diff.) identities:

$$\begin{aligned} \sin(t+h) - \sin(t-h) &= 2\cos t \sin h \\ \cos(t+h) - \cos(t-h) &= -2\sin t \sin h \end{aligned} \quad (**)$$

follows

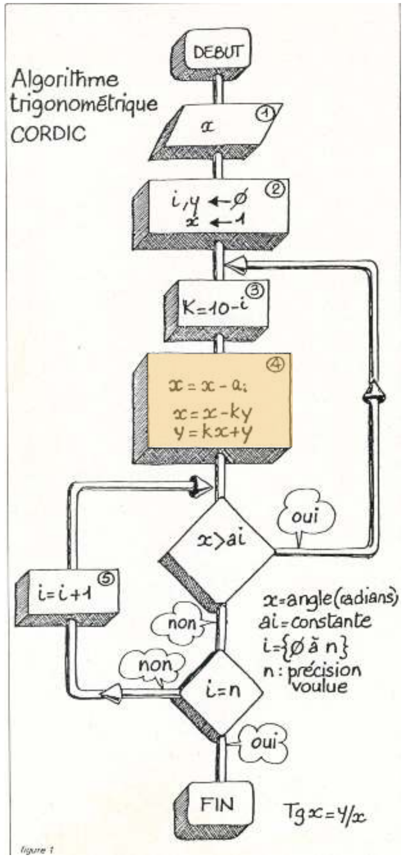
$$\begin{aligned} \cos((t+h)+h) - \cos((t+h)-h) &= -2\sin(t+h)\sin h = -K\sin(t+h) \\ \sin((t+2h)+h) - \sin((t+2h)-h) &= 2\cos(t+2h)\sin h = K\cos(t+2h) \end{aligned}$$

Thus a table of trigonometric functions is calculated by an initialisation (calculate $\cos t, \sin(t+h), K = 2\sin h$) and by 1 addition and 1 multiplication (with a suitable K as binary shift) for each step h:

$$\begin{aligned} \cos(t+2h) &= \cos(t) - K\sin(t+h) \\ \sin(t+3h) &= \sin(t+h) + K\cos(t+2h) \\ \cos(t+4h) &= \cos(t+2h) - K\sin(t+3h) \\ \text{etc} & \quad \dots \end{aligned}$$

Scheme:
$(x_i := \cos(t+i\,h), y_i := \sin(t+i\,h))$

$$\begin{aligned} x_0 &= \cos(0) = 1 \\ y_1 &= \sin(h) \\ K &= 2\sin(h) = 2y_1 \\ x_2 &= x_0 - Ky_1 \\ y_3 &= y_1 + Kx_2 \\ x_4 &= x_2 - Ky_3 \\ &\dots \end{aligned}$$



(Laporte 1981)

Iterative Functions IV (1959)

"the barycentric method": subdivide a given polynomial


P2108_9

$G = \dfrac{pA + qB}{p+q}$ barycentre of points A, B with masses p, q → iterative evaluation of point M,

Use $p = q = \dfrac{1}{2}$: subdivided control polygons $AGIM$ and $MJKD$ (de Casteljau 1959).

Thus, the values of a cubic polynomial $P(t), t \in [0,1]$, that is represented by a control polygon $ABCD$, are iteratively calculated by $n(n+1) = 12$ multiplications and $\dfrac{n(n+1)}{2} = 6$ additions.

Remark: An upfront calculation of Bernstein polynomials reduces it to $(n+1) = 4$ multiplications and $n = 3$ additions.

The iterative approach also delivers tangent $dM = 3(I - J)\dfrac{dp}{p+q}$ and curvature radius $R_A = \dfrac{3}{2}\dfrac{\overline{AB}^2}{\overline{HC}}$ at M (where H is foot of the altitude from C to AB).

Triangular Scheme:

. With $\alpha = \dfrac{p}{p+q} = (1-t)$ and $\beta = \dfrac{q}{p+q} = t$, we get:

$$
\begin{array}{cccc}
A & & & \\
 & G = \alpha A + \beta B & & \\
B & & I = \alpha G + \beta H & \\
 & H = \alpha B + \beta C & & M = \alpha I + \beta J \\
C & & J = \alpha H + \beta K & \\
 & K = \alpha C + \beta D & & \\
D & & &
\end{array}
$$

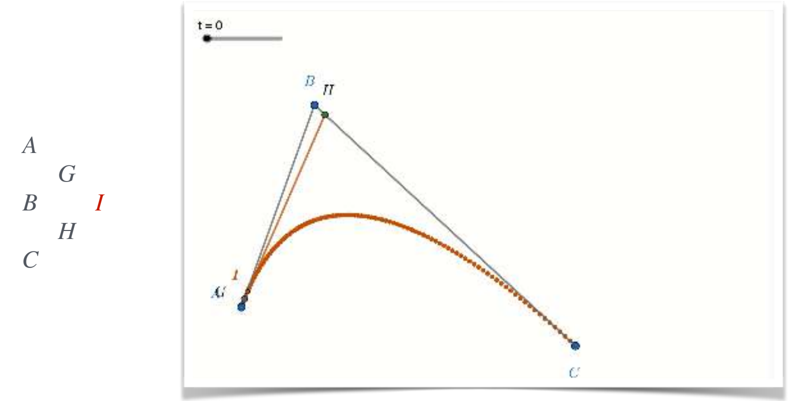
"The method of tracing curves by points has been created to satisfy the needs of practice; it already appears so interesting itself that it could establish itself even without the support of the computation method." (de Casteljau 1959)

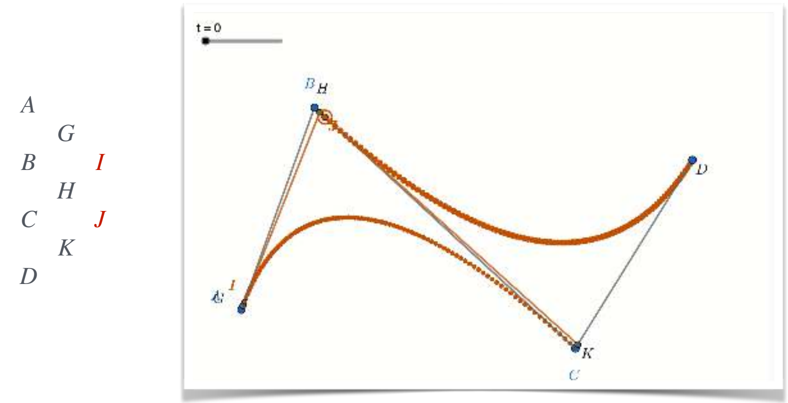
Iterative Functions V (1959)

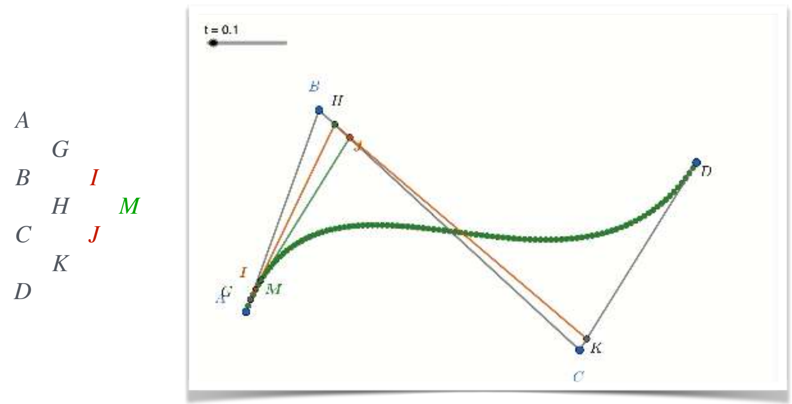
"the barycentric method": genesis

Three different hypotheses for de Casteljau's "DIY" approach in 1959:

A. (arithmetic) Take iterative function I and replace the difference $P_1 - P_0$ by an addition $\alpha P_0 + \beta P_1$ (Rabut 2024)
B. (geometric) Construct a parabola's point and tangent from 3 "poles" and 2+1 points, generalise to $n = 3$.
C. (numerical) Take iterative function I and evaluate $t = \dfrac{k}{h} \in [0,1]$ instead of $t = k \cdot h \in [0,\infty]$





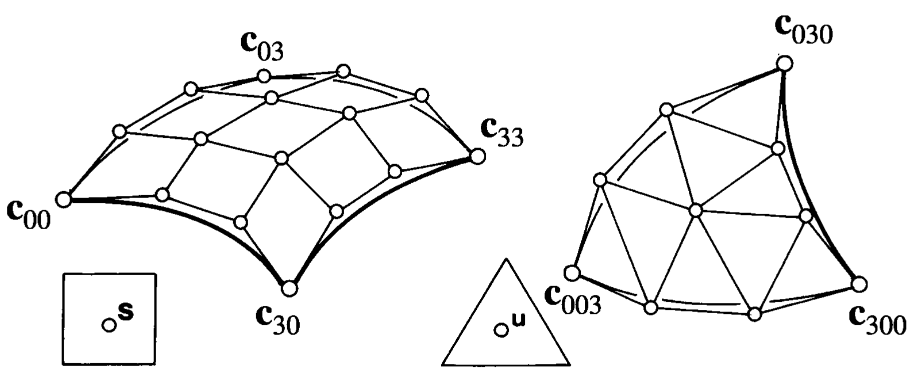


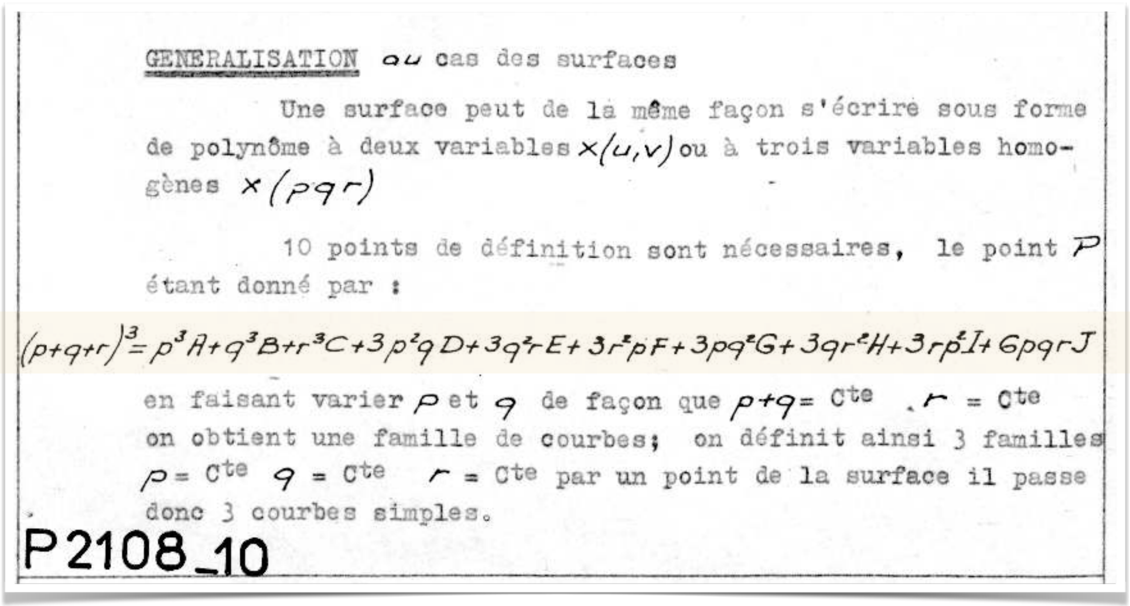
Triangular Surfaces (already 1959)

(1963 tensor product surfaces)

Together with 10 points $A, \ldots, J$, the barycentric (triangular) coordinates p, q, r define a point P of (triangular) surface:

$$(p+q+r)^3P = p^3A + q^3B + r^3C + 3p^2qD + 3q^2rE + 3r^2pF + 3pq^2G + 2qr^2H + 3rp^2I + 6pqrJ$$





GENERALISATION au cas des surfaces

Une surface peut de la même façon s'écrire sous forme de polynôme à deux variables $x(u,v)$ ou à trois variables homogènes $x(pqr)$

10 points de définition sont nécessaires, le point P étant donné par :

$$(p+q+r)^3 = p^3A+q^3B+r^3C+3p^2qD+3q^2rE+3r^2pF+3pq^2G+3qr^2H+3rp^2I+6pqrJ$$

en faisant varier p et q de façon que $p+q=$ Cte . r = Cte on obtient une famille de courbes; on définit ainsi 3 familles $p=$ Cte q = Cte r = Cte par un point de la surface il passe donc 3 courbes simples.

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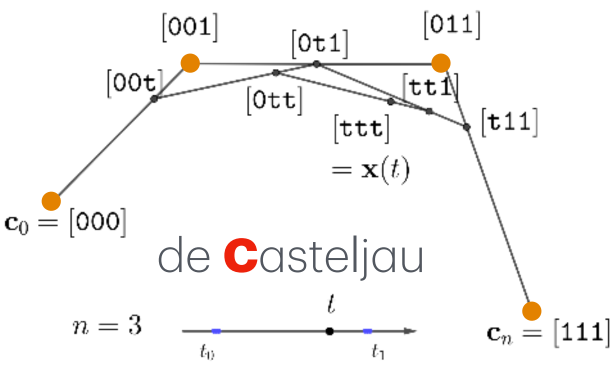
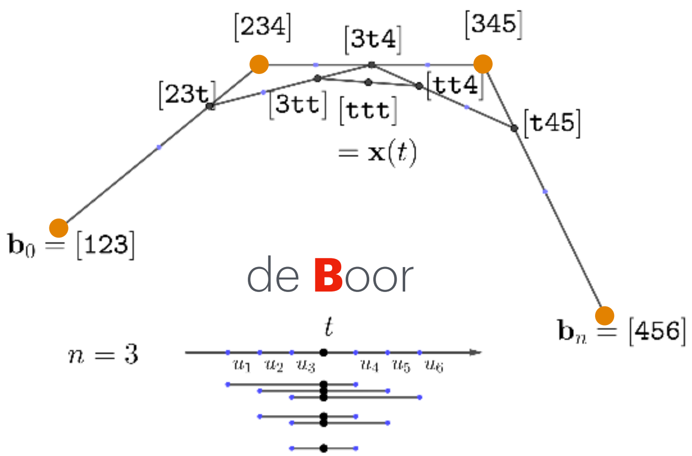
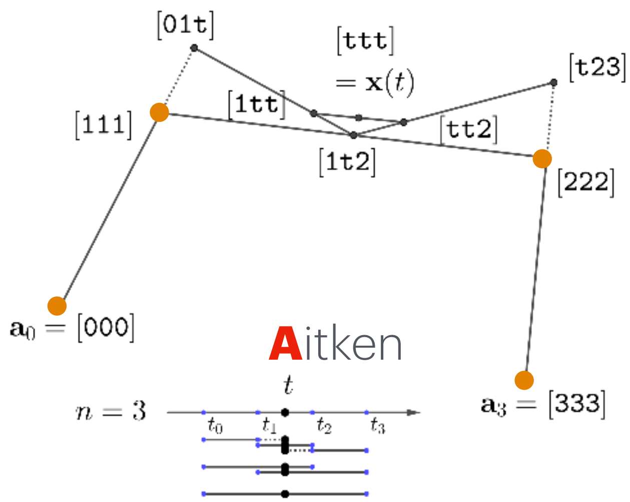
Polar Forms (1983, 1986)

aka. Blossoming

$\mathbf{x}[t_1, \ldots, t_n] := \sum_{i=0}^{n} \mathbf{p}_i \sigma_i^n$ is the polar form of the polynomial curve $\mathbf{x}(t) = \sum_{i=0}^{n} \mathbf{p}_i \binom{n}{i} t^n$, with elementary symmetric functions $\sigma_i^n = \sigma_i(t, t, \ldots, t) = \binom{n}{i} t^i$. This approach allows to display a *curve* and its *control points* in an integral manner. Obviously, $\mathbf{x}[t, \ldots, t] = \mathbf{x}(t)$ represents the curve itself, furthermore some parameters represent interesting control points:







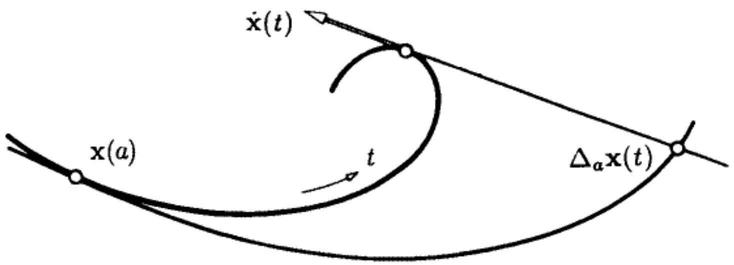
Index reduction

All three algorithms are based on the multi affinity of a polar form:

$$
\begin{aligned}
x[\underbrace{t_1, \cdots, t_{n-1}}_{n-1}](t) &= x[t_1, \cdots, t_{n-1}](t_0) \cdot (1-\alpha) + x[t_1, \cdots, t_{n-1}](t_n) \cdot \alpha \\
&= x[\underbrace{t_0, \cdots, t_{n-1}}_{n}] \cdot (1-\alpha) + x[\underbrace{t_1, \cdots, t_n}_{n}] \cdot \alpha
\end{aligned}
\qquad \text{mit} \quad \alpha = \frac{t_0 - t}{t_0 - t_n}
$$

where $x[t_1, \cdots, t_{n-1}](t)$ depends on $(n-1)$ variables, while $x[t_0, \cdots, t_{n-1}]$ as well as $x[t_1, \cdots, t_n]$ are defined by n variables. The evaluation of t reduces the polar form by one index/variable.

Remark: Geometrically, this is related to the reduced degree of the osculant: $\Delta_a \mathbf{x}(t) = \mathbf{x}(t) + \frac{a-t}{n} \cdot \dot{\mathbf{x}}(t)$
(Haase, 1870)



(Boehm/Prautzsch, 1994)

Algebraic Smoothing

“The” algorithm
Algebraic smoothing
Metric geometry:
    focal splines, 14-point strophoid, conjugate mirrors, regular polygons
16D+3C

Algebraic Smoothing (1983, 1986)

"interpolation par polynômes écrits sous forme polaire" → *"formes à pôles"*

In three steps, we will generate the (Bézier/de Casteljau) control points for some curve segments of degree n, which smoothly (C^k) connect q given points, that we assume to lie on curves of (sampling) degree r, respectively. Let's consider the example $(n, k, r) = (5,3,3)$ with $q = 6$:

1. Lagrange interpolation with *restititutional* degree r (delivering $\mathbf{a}_i$),

   E.g., $r = 3$: with $q = 6$ given points $\mathbf{a}_{-2}, \cdots, \mathbf{a}_3$, we have $q - r = 3$ polynomial each interpolating $r + 1 = 4$ points,

   L_{-2-101} • • • •

   also L_{-1012} • • • •

   L_{0123} • • • •

2. Polar form of degree n with C^k-continuity (delivering $\mathbf{b}_j$) by $(n - k)$-fold indexes (*knots*),

   E.g., $n - k = 2$, thus 2-fold indexes, which leads to the following polar forms of degree $n = 5$:

   [-2 -1 -1 0 0], [-1 -1 0 0 1], [-1 0 0 1 1], [0 0 1 1 2], [0 1 1 2 2], [1 1 2 2 3]

3. Bézier curve (delivering $\mathbf{c}_k$) by index repetition (*knot insertion*)

(de Casteljau 1983)

The configuration $(n = 5, k = 3, r = 3)$

$$
\begin{array}{rccccl}
 & x^3 & x^2 & x^1 & x^0 & \\
\div (x+1): & 1 & -3 & \mathbf{2} & 0 & [\div -\mathbf{6}] \\
\div x: & 1 & -2 & -1 & 2 & [\div +2] \\
\div (x-1): & 1 & -1 & -2 & 0 & [\div -2] \\
\div (x-2): & 1 & 0 & -1 & 0 & [\div +6] \\
 & [\div 10] & [\div 10] & [\div \mathbf{5}] & [\div 1] & \\
 & \sigma_3^5 & \sigma_2^5 & \sigma_1^5 & 1 & \\
L_{-1012}[-1,0,0,1,1]: & -1 & -1 & \mathbf{1} & 1 & \\
L_{-1012}[0,0,1,1,2]: & 2 & 5 & 4 & 1 &
\end{array}
$$

$$\rightarrow L_{-1012}(x) = \frac{x^3 - 3x^2 + 2x}{-6}\mathbf{a}_{-1} + \frac{x^3 - 2x^2 - x + 2}{2}\mathbf{a}_0 + \frac{x^3 - x^2 - 2x}{-2}\mathbf{a}_1 + \frac{x^3 - x}{6}\mathbf{a}_2$$

$$
\begin{aligned}
L_{-1012}[-1,0,0,1,1] &= \left(\frac{1}{1}\cdot\frac{0}{-6} + \frac{\mathbf{1}}{\mathbf{5}}\cdot\frac{\mathbf{2}}{\mathbf{-6}} + \frac{1}{10}\cdot\frac{-3}{-6} + \frac{-1}{10}\cdot\frac{1}{-6}\right)\mathbf{a}_{-1} + \left(\frac{2}{2} + \frac{-1}{10} + \frac{2}{20} + \frac{-1}{20}\right)\mathbf{a}_0 \\
&\quad + \left(\frac{0}{-2} + \frac{-2}{-10} + \frac{1}{-20} + \frac{-1}{-20}\right)\mathbf{a}_1 + \left(\frac{0}{6} + \frac{-1}{30} + \frac{0}{60} + \frac{-1}{60}\right)\mathbf{a}_2 \\
&= \frac{1}{60}[-6 \quad 57 \quad 12 \quad -3]\,[\mathbf{a}_{-1} \quad \mathbf{a}_0 \quad \mathbf{a}_1 \quad \mathbf{a}_2] \\
L_{-1012}[0,0,1,1,2] &= \frac{1}{60}[-3 \quad 12 \quad 57 \quad -6]\,[\mathbf{a}_{-1} \quad \mathbf{a}_0 \quad \mathbf{a}_1 \quad \mathbf{a}_2]
\end{aligned}
$$

$$
\begin{bmatrix} \mathbf{b}_{-2} \\ \mathbf{b}_{-1} \\ \mathbf{b}_0 \\ \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{bmatrix}
=
\begin{bmatrix} L_{-2-101}[-2,-1,-1,0,0] \\ L_{-2-101}[-1,-1,0,0,1] \\ L_{-1012}[-1,0,0,1,1] \\ L_{-1012}[0,0,1,1,2] \\ L_{0123}[0,1,1,2,2] \\ L_{0123}[1,1,2,2,3] \end{bmatrix}
= \frac{1}{60}
\begin{bmatrix}
-6 & 57 & 12 & -3 & & \\
-3 & 12 & 57 & -6 & & \\
 & -6 & 57 & 12 & -3 & \\
 & -3 & 12 & 57 & -6 & \\
 & & -6 & 57 & 12 & -3 \\
 & & -3 & 12 & 57 & -6
\end{bmatrix}
\begin{bmatrix} \mathbf{a}_{-2} \\ \mathbf{a}_{-1} \\ \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix}
$$

$$
\begin{bmatrix} \mathbf{-2, -1, -1, 0, 0} \\ \mathbf{-1, -1, 0, 0, 1} \\ -1, 0, 0, 1, 1 \\ 0, 0, 1, 1, 2 \\ 0, 1, 1, 2, 2 \\ 1, 1, 2, 2, 3 \end{bmatrix}
\xrightarrow{K_1}
\begin{bmatrix} \mathbf{-1, -1, 0, 0, 0} \\ -1, 0, 0, 0, 1 \\ 0, 0, 0, 1, 1 \\ 0, 0, 1, 1, 1 \\ 0, 1, 1, 1, 2 \\ 1, 1, 1, 2, 2 \end{bmatrix}
\xrightarrow{K_2}
\begin{bmatrix} -1, 0, 0, 0, 0 \\ 0, 0, 0, 0, 1 \\ 0, 0, 0, 1, 1 \\ 0, 0, 1, 1, 1 \\ 0, 1, 1, 1, 1 \\ 1, 1, 1, 1, 2 \end{bmatrix}
\xrightarrow{K_3}
\begin{bmatrix} 0, 0, 0, 0, 0 \\ 0, 0, 0, 0, 1 \\ 0, 0, 0, 1, 1 \\ 0, 0, 1, 1, 1 \\ 0, 1, 1, 1, 1 \\ 1, 1, 1, 1, 1 \end{bmatrix}
\rightarrow
$$

$$
\begin{bmatrix} \mathbf{c}_0 \\ \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{c}_4 \\ \mathbf{c}_5 \end{bmatrix}
= \frac{1}{2}
\begin{bmatrix}
1 & 1 & & & & \\
 & 2 & & & & \\
 & & 2 & & & \\
 & & & 2 & & \\
 & & & & 2 & \\
 & & & & 1 & 1
\end{bmatrix}
\cdot \frac{1}{2}
\begin{bmatrix}
1 & 1 & & & & \\
 & 1 & 1 & & & \\
 & & 2 & & & \\
 & & & 2 & & \\
 & & & 1 & 1 & \\
 & & & & 1 & 1
\end{bmatrix}
\cdot \frac{1}{6}
\begin{bmatrix}
2 & 4 & & & & \\
 & 3 & 3 & & & \\
 & & 4 & 2 & & \\
 & & 2 & 4 & -6 & \\
 & & & 3 & 3 & \\
 & & & & 4 & 2
\end{bmatrix}
\begin{bmatrix} \mathbf{b}_{-2} \\ \mathbf{b}_{-1} \\ \mathbf{b}_0 \\ \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{bmatrix}
= \frac{1}{240}
\begin{bmatrix}
-7 & 28 & 198 & 28 & -7 & \\
-3 & -4 & 198 & 60 & -11 & \\
 & -20 & 168 & 108 & -16 & \\
 & -16 & 108 & 168 & -20 & \\
 & -11 & 60 & 198 & -4 & -3 \\
 & -7 & 28 & 198 & 28 & -7
\end{bmatrix}
\begin{bmatrix} \mathbf{a}_{-2} \\ \mathbf{a}_{-1} \\ \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix}
$$

The configuration $(n = 5, k = 2, r = 4)$

	x^5	x^4	x^3	x^2	x^1	x^0	$\frac{1}{24}\times$
$L_{-2-1012}:$	1	0	-5	0	4	0	
$\div(x+2):$		1	-2	$\mathbf{-1}$	$\mathbf{2}$	0	$[+\mathbf{1}]$
$\div(x+1):$		1	-1	-4	4	0	$[-4]$
$\div x:$		1	0	-5	0	4	$[+6]$
$\div(x-1):$		1	1	-4	-4	0	$[-4]$
$\div(x-2):$		1	2	-1	-2	0	$[+1]$
$\frac{1}{10}\times$		[2]	[1]	$[\mathbf{1}]$	$[\mathbf{2}]$	[10]	
		σ_4^5	σ_3^5	σ_2^5	σ_1^5	1	
$L_{-2-1012}[-1,-1,0,0,0]:$		0	0	$\mathbf{1}$	$\mathbf{-2}$	1	
$L_{-2-1012}[-1,0,0,0,1]:$		0	0	-1	0	1	
$L_{-2-1012}[0,0,0,1,1]:$		0	0	1	2	1	

$$\to l_{-2} = \frac{(x+1)x(x-1)(x-2)}{(-2-(-1))(-2-0)(-2-1)(-2-2)}$$

$$\to \frac{1}{10}\frac{1}{24}(-2\cdot 2\cdot 2\cdot 1 + 1\cdot 1\cdot(-1)\cdot 1)\,\mathbf{a}_{-2} + \ldots$$

$$\begin{bmatrix}\mathbf{b}_{-2}\\ \mathbf{b}_{-1}\\ \mathbf{b}_0\\ \mathbf{b}_1\\ \mathbf{b}_2\\ \mathbf{b}_3\end{bmatrix} = \frac{1}{240}\begin{bmatrix}-9 & 80 & 210 & -48 & 7 & \\ 1 & -16 & 270 & -16 & 1 & \\ 7 & -48 & 210 & 80 & -9 & \\ & -9 & 80 & 210 & -48 & 7\\ & 1 & -16 & 270 & -16 & 1\\ & 7 & -48 & 210 & 80 & -9\end{bmatrix}\begin{bmatrix}\mathbf{a}_{-2}\\ \mathbf{a}_{-1}\\ \mathbf{a}_0\\ \mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\end{bmatrix}$$

$$\begin{bmatrix}\mathbf{c}_0\\ \mathbf{c}_1\\ \mathbf{c}_2\\ \mathbf{c}_3\\ \mathbf{c}_4\\ \mathbf{c}_5\end{bmatrix} = \frac{1}{4\cdot 240}\begin{bmatrix}1 & 2 & 1 & & & \\ & 2 & 2 & & & \\ & & 4 & & & \\ & & & 4 & & \\ & & & 2 & 2 & \\ & & & 1 & 2 & 1\end{bmatrix}\begin{bmatrix}-9 & 80 & 210 & -48 & 7 & \\ 1 & -16 & 270 & -16 & 1 & \\ 7 & -48 & 210 & 80 & -9 & \\ & -9 & 80 & 210 & -48 & 7\\ & 1 & -16 & 270 & -16 & 1\\ & 7 & -48 & 210 & 80 & -9\end{bmatrix}\begin{bmatrix}\mathbf{a}_{-2}\\ \mathbf{a}_{-1}\\ \mathbf{a}_0\\ \mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\end{bmatrix} = \frac{1}{240}\begin{bmatrix} & & 240 & & & \\ 4 & -32 & 240 & 32 & -4 & \\ 7 & -48 & 210 & 80 & -9 & \\ & -9 & 80 & 210 & -48 & 7\\ & -4 & 32 & 240 & -32 & 4\\ & & & 240 & & \end{bmatrix}\begin{bmatrix}\mathbf{a}_{-2}\\ \mathbf{a}_{-1}\\ \mathbf{a}_0\\ \mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\end{bmatrix}$$

The configuration $(n = 5, k = 4, r = 1)$

$$
\begin{array}{rcccc}
 & x^2 & x^1 & x^0 & \frac{1}{1}\times \\
L_{01}: & 1 & -1 & 0 & \\
\div(x): & & 1 & -1 & [-1] \\
\div(x-1): & & 1 & 0 & [+1] \\
\frac{1}{5}\times & & [1] & [5] & \\
 & & \sigma_1^5 & 1 & \\
L_{01}[-2,-1,0,1,2]: & & 0 & 1 &
\end{array}
$$

$$\rightarrow l_0 = \frac{(x-1)}{(0-1)} = 1 - x$$

$$\rightarrow l_1 = \frac{(x)}{(1-0)} = x$$

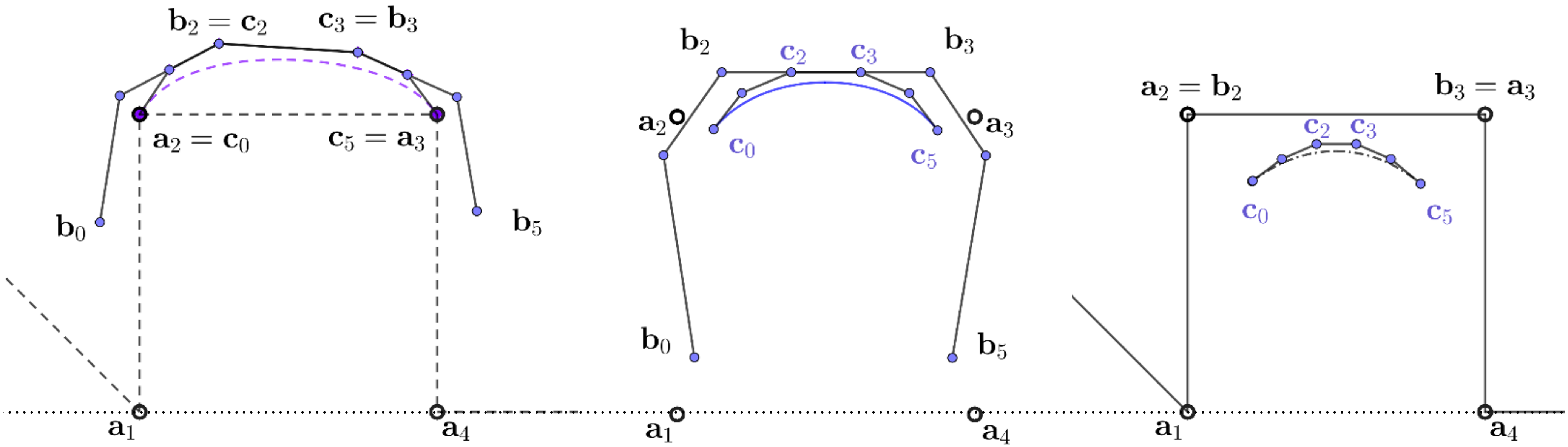
$$\rightarrow \frac{1}{5}\frac{1}{1}(1 \cdot 5 \cdot (-1) \cdot (-1))\,\mathbf{a}_0$$

$$
\begin{bmatrix} \mathbf{b}_{-2} \\ \mathbf{b}_{-1} \\ \mathbf{b}_0 \\ \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{bmatrix}
=
\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}
\begin{bmatrix} \mathbf{a}_{-2} \\ \mathbf{a}_{-1} \\ \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix}
$$

$$
\begin{bmatrix} \mathbf{c}_0 \\ \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{c}_4 \\ \mathbf{c}_5 \end{bmatrix}
= \frac{1}{120}
\begin{bmatrix} 1 & 26 & 66 & 26 & 1 & \\ & 16 & 66 & 36 & 2 & \\ & 8 & 60 & 48 & 4 & \\ & 4 & 48 & 60 & 8 & \\ & 2 & 36 & 66 & 16 & \\ & 1 & 26 & 66 & 26 & 1 \end{bmatrix}
\begin{bmatrix} \mathbf{a}_{-2} \\ \mathbf{a}_{-1} \\ \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix}
$$

Algebraic Smoothing

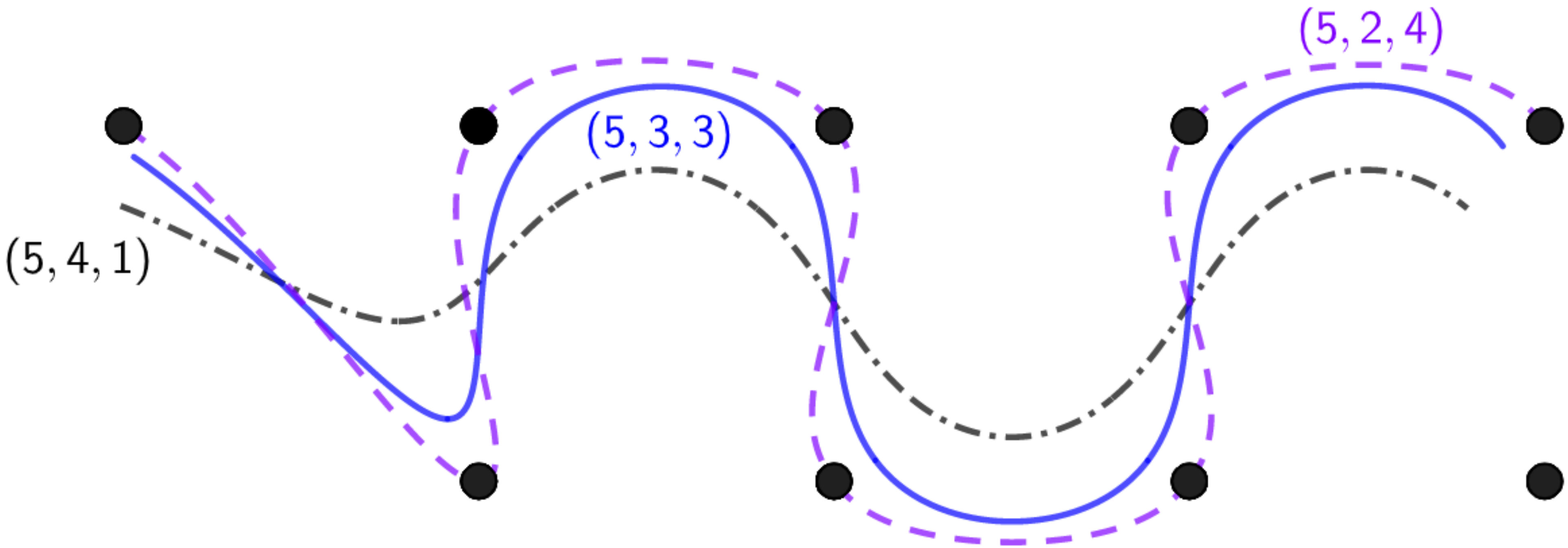
(degree n, continuity k, restitution r)



(5,2,4) (5,3,3) (5,4,1)

Algebraic Smoothing

high continuity (5,4,1) vs. high restitution (5,2,4)



Algebraic Smoothing

Optimal Smoothing, comparison with B-Splines

Compare the algebraic conditions for continuity and restitution with the transformation matrix. Optimal is the minimal no of coefficients per no of conditions.

Example $n = 11$:

config	q	coeff's: $q(n+1)$	cond	
$(11,10,11)$	22	$12 \cdot 22 = 264$	397	$264/397 = 66{,}5\,\%$
$(11,5,10)$	12	$12 \cdot 12 = 144$	210	$144/210 = 68{,}6\,\%$
$(11,8,10)$	14	$12 \cdot 14 = 168$	267	$168/267 = \mathbf{62,9}\,\%$

→ high continuity

(n,k,r)	$s = 2$ segments	3	4	5	6
$q = 4$ points	(3,1,2)	(3,1,1) CR spline	(3,2,1) **B-spline**		
6	$(\mathbf{5,2,4})$	$(\mathbf{5,3,3})$	(3,2,3) **B-spline**	(4,3,1) **B-spline**	$(\mathbf{5,4,1})$ **B-spline**
8	(7,3,6)	(5,3,5)	(7,5,4)	(4,3,3) **B-spline**	(5,4,3) **B-spline**
10	(9,4,8)	(8,5,7)	(7,5,6)	(9,7,5)	(5,4,5) **B-spline**
12		(11,7,9)	(11,8,8)	(9,7,7)	(11,9,6)
14	↑	(11,7,11)	(11,8,10)	(9,7,9)	(11,9,8)
16	Riabenki	(14,9,13)	(15,11,12)	(14,11,11)	(11,9,10)
18	↓	(17,11,15)	(15,11,14)	(14,11,13)	(17,14,12)
20		(17,11,17)	(19,14,16)	(19,15,15)	(17,14,14)

↓ high restitution

de Casteljau's four principles

for working with curves and surfaces

1. To avoid the infinite and cyclical points, ALL intermediary construction points must be **on the piece of paper**, which condemns rational splines (unless weights are composed of squares, therefore positive)

2. The tool works only on **one point at a time**. That makes the notion of a parameter seem logical, as you do not have to choose between multiple solutions.

3. Only the **real domain** is accessible. All passage through complex concepts remains dangerous.

4. All too often, the physical nature of mathematisation is forgotten, especially the concept of '**error calculation**'. It is advisable to avoid indeterminate forms and leave them to Calculus, e.g., the differences of two very large numbers or the quotients of two small quantities. These considerations are well disguised when it comes to matrix calculus or least squares, which have the appearance of confidence-inspiring accuracy while approaching "overflows" or "underflows". I can't warn enough against giving in to the illusion of getting everything right while acrobatically teetering on the edges of the abyss.

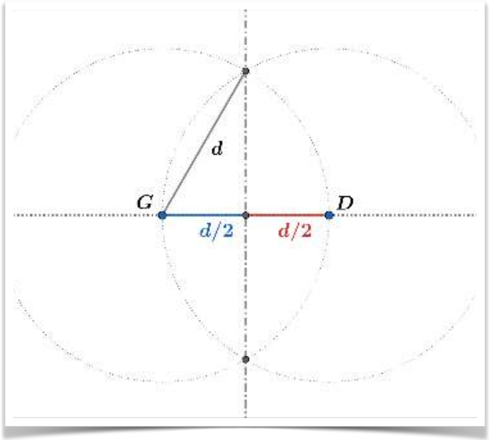
(de Casteljau 2006)

Metric Geometry

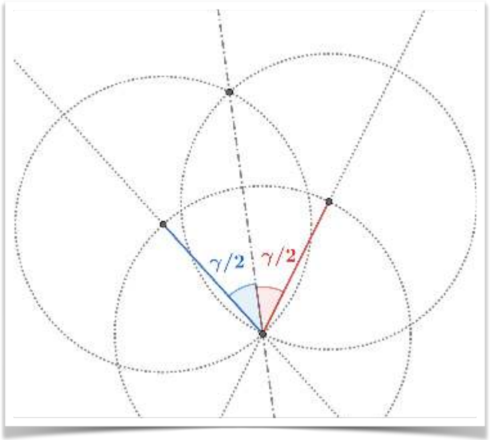
“The” algorithm
Algebraic smoothing
Metric geometry:
    focal splines, 14-point strophoid, conjugate mirrors, regular polygons
16D+3C

Metric Geometry (1994, 1995)

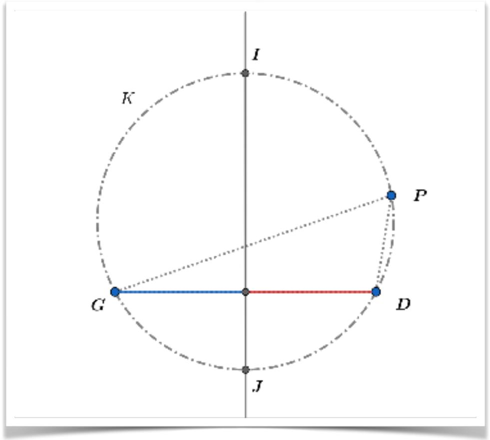
Dualities



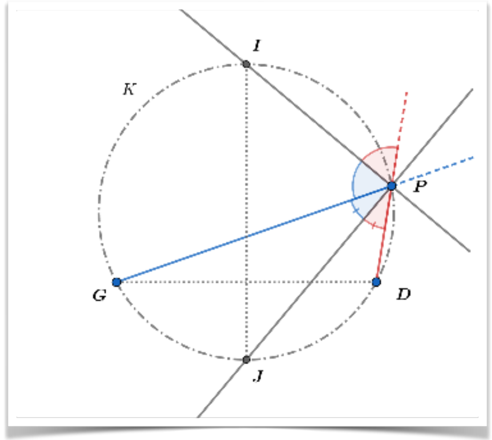
2 points define their **distance**,
cut into two perpendicular bisectors



2 lines define their **angle**,
cut into two angle bisectors



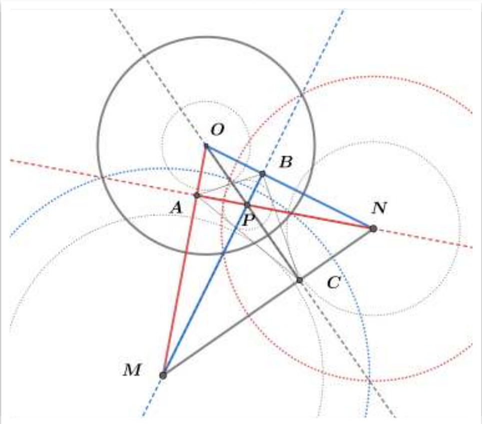
symmetry axis of GD is **perpendicular bisector** IJ, which halvens distance GD.

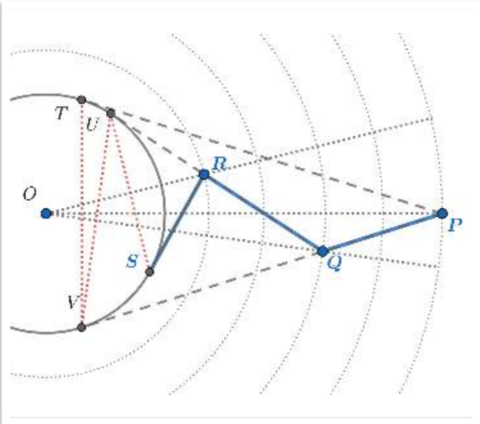


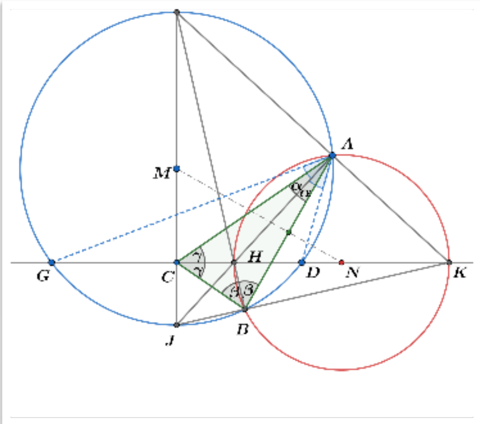
symmetry axes of GP, PD are **angle bisectors** PI, PJ, which halven angle GPD.

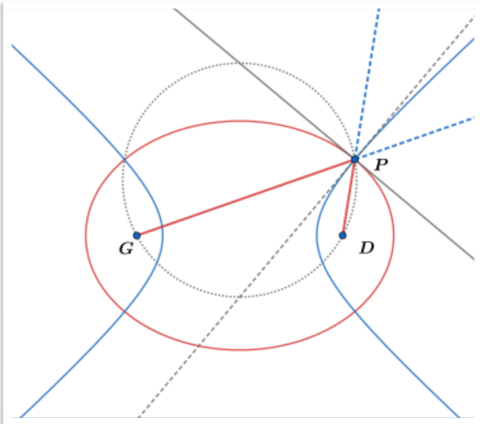
“Her Majesty, the Circle” (1995)

and Their Highnesses the Focal Conics - and not just curves of second degree









Orthocentric tetragram:

Each of the points O, M, N, P is orthocenter of the triangle of the remaining three points.

O, M, N, P are also centers of the ex-/inscribed circles to orthic triangle ABC

The tangential distance PT to the circle is equal to the reflection line $PQRS$

Apollonian (orthogonal) circles:

'elliptic' circle passing through G, D and 'hyperbolic' circle limited at G, D

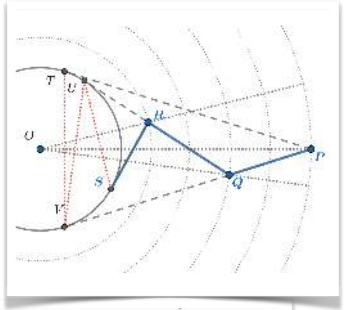
The tangent to the ellipse in P is the normal of the confocal hyperbola.

The tangent to the hyperbola in P is the normal of the confocal ellipse

Focal Splines

“The” algorithm
Algebraic smoothing
Metric geometry:
    focal splines, 14-point strophoid, conjugate mirrors, regular polygons
16D+3C

Focal Splines (1994)



The tangential distance of a point G to a circle is equal to the reflection lines $Gi = GH_0 j = GE_4 j$. Thus

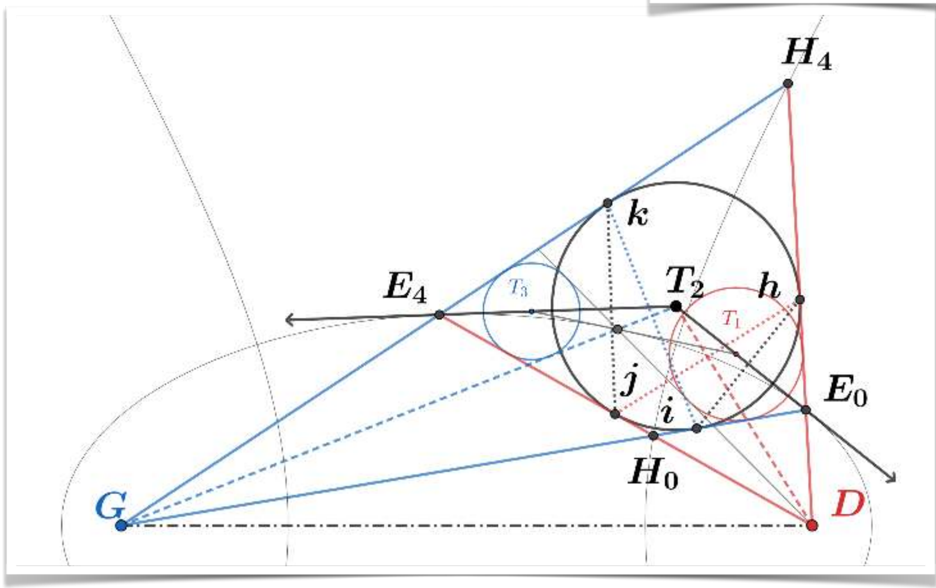
$$\begin{aligned} Gi + jD &= Gk + jD = GE_4 + E_4 j + jD &&= GE_4 + E_4 D \\ &= Gi + hD = Gi + iE_0 + E_0 D &&= GE_0 + E_0 D \end{aligned}$$

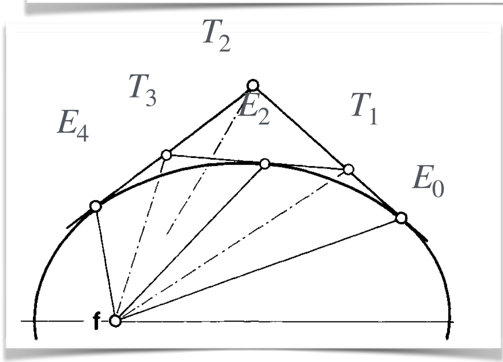
$$\begin{aligned} Gk - jD &= Gk + (kH_4 - H_4 h) - hD &&= GH_4 - H_4 D \\ &= Gi - jD + (jH_0 - H_0 i) &&= GH_0 - H_0 D \end{aligned}$$

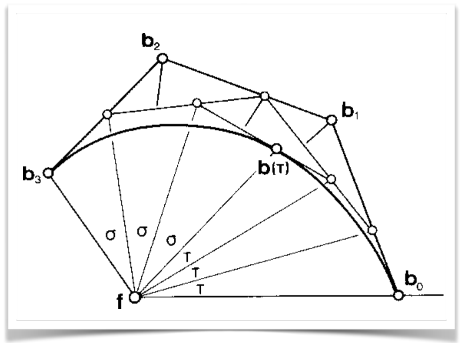
So, points E_i lie on an ellipse and points H_i on a hyperbola.

Even more: as E_4T_2 is tangent, we can subdivide and find a circle around T_3 at some angle $\sigma : \tau$, which leads to E_2 on the ellipse. This forms a recursion and can be generalised to splines via polar forms.

(remark: think of compound trajectories, e.g., moon-earth-sun)







(de Casteljau 1994, Boehm/Müller 1999)

14-point strophoid

“The” algorithm
Algebraic smoothing
Metric geometry:
    focal splines, 14-point strophoid, conjugate mirrors, regular polygons
16D+3C

De Casteljau's 14 point strophoid

Theorem (14 point strophoid)

Given a complete quadrilateral, then 14 points lie on a strophoid:

- 1 center T of the incircle to a complete quadrilateral (counts twice)
- 6 intersections of the complete quadrilateral A_1, B_1, A_2, B_2, A_3, B_3
- 3 feet H_1, H_2, H_3 of triangles A_iB_iT
- 3 intersections K_1, K_2, K_3 of an A_iB_i parallel through T with perp. bisector

Theorem (9 point circle)

Given a triangle, then 9 points lie on a circle:

- 3 midpoints of each side of the triangle
- 3 feet H_1, H_2, H_3 of the triangle's altitudes
- 3 midpoints of the line segment between each vertex and the orthocenter

De Casteljau's 14 point strophoid

we start with a Complete Quadrilateral

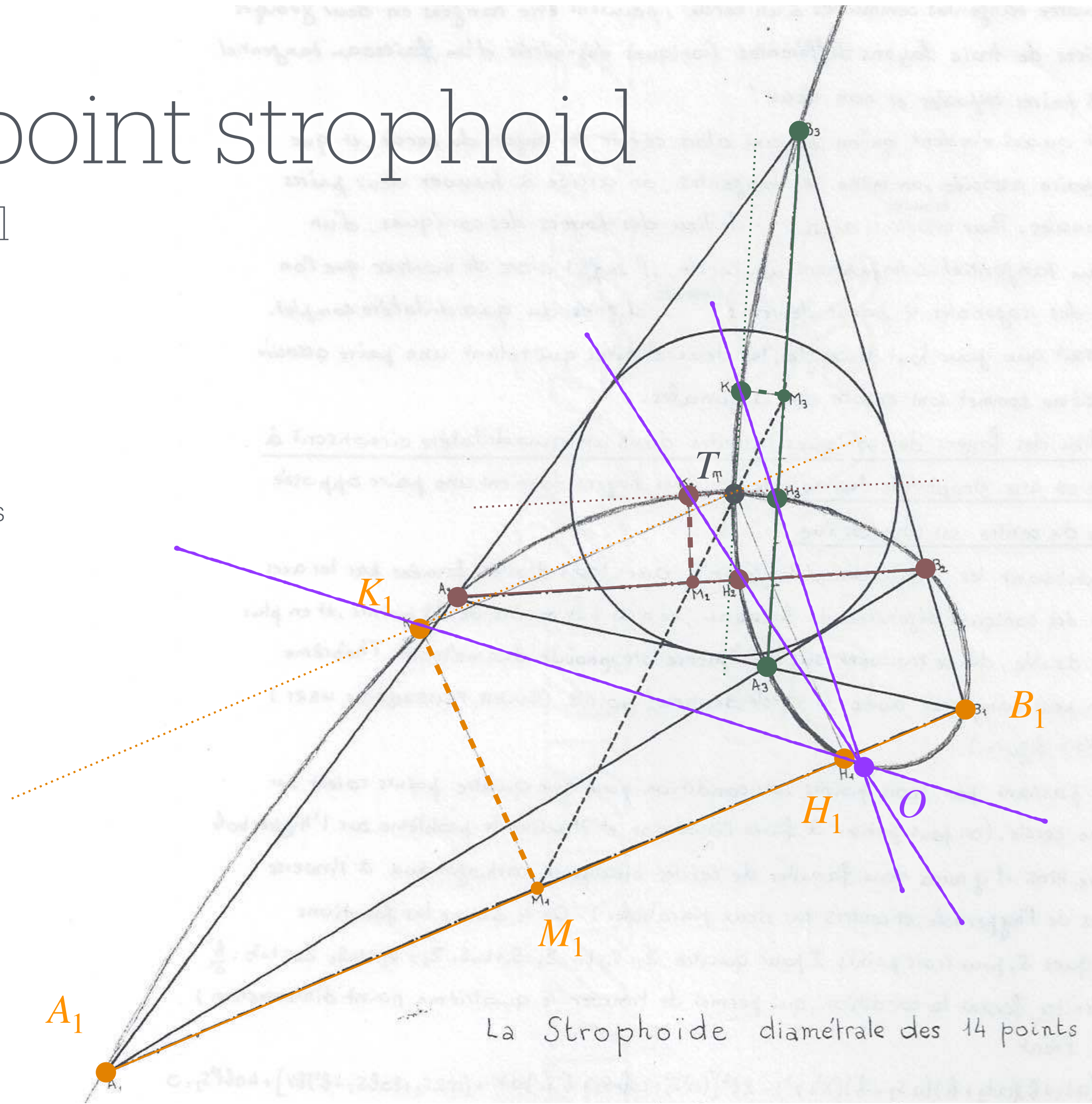
Theorem (14 point strophoid)

14 points lie on a strophoid:

- 1 center T of the incircle to a complete quadrilateral (counts twice)
- 6 intersections of the complete quadrilateral $A_1, B_1, A_2, B_2, A_3, B_3$
- 3 feet H_1, H_2, H_3 of triangles A_iB_iT
- 3 intersections K_1, K_2, K_3 of an A_iB_i parallel through T with perp. bisector

Even more:

H_1K_1, H_2K_2, H_3K_3 meet in O on the strophoid (point 15).



Conjugate Mirrors

“The” algorithm
Algebraic smoothing
Metric geometry:
- focal splines, 14-point strophoid, conjugate mirrors, regular polygons

16D+3C

Conjugate Mirrors (1974)

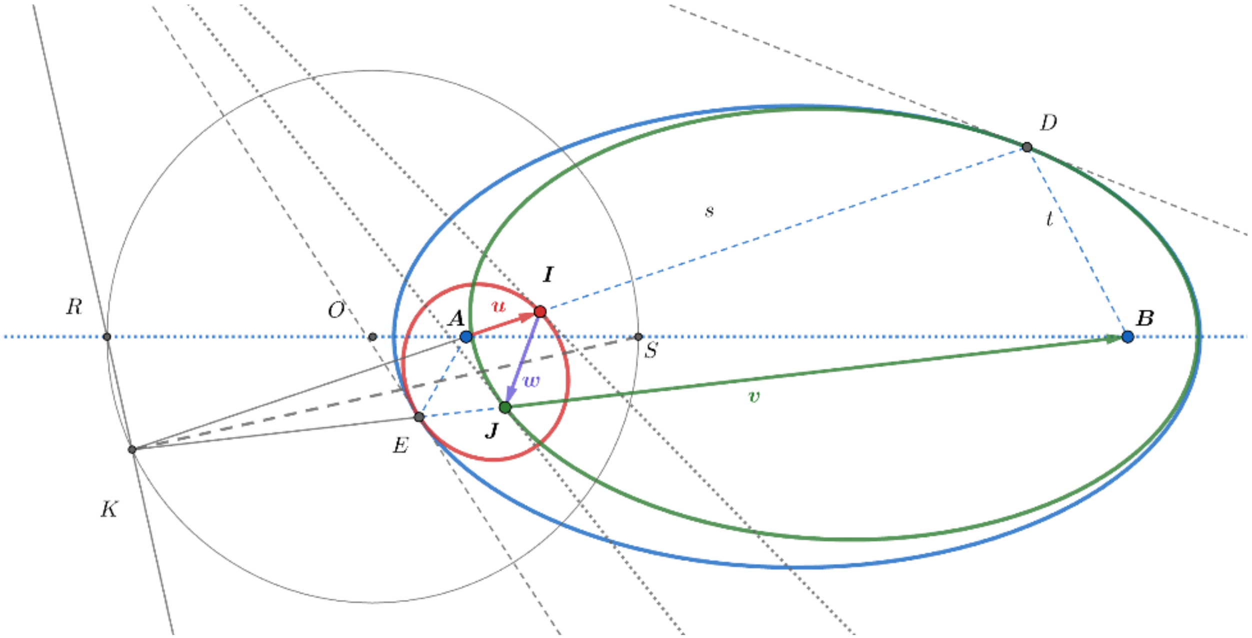
A mirror/string construction with $k > 1$ degrees of freedom

A wavefront from B to A has same distances (Fermat) via:

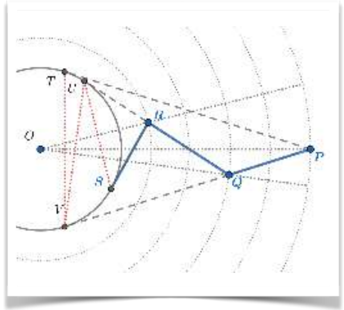
$$\begin{aligned} & BDA &=& \quad BDIA \\ = \; & BEA &=& \quad BJEA \\ = \; & BJIA &=& \quad u + v + w = 2a \end{aligned}$$

I.e., instead of one mirror at D, we can put two conjugate mirrors at I and J, whose tangents meet those at D and E in a point $T_{rr'}$.

Even more, the perpendicular bisector of the intersection K of both rays (AI and BJ) is perpendicular to $KT_{rr'}$ and K lies on a (Thales) circle symmetric to the main axis AB, which meets KT_{rr} in R.



Conjugate Mirrors



From $IJ = kJ - kI = iJ - jI$ follows

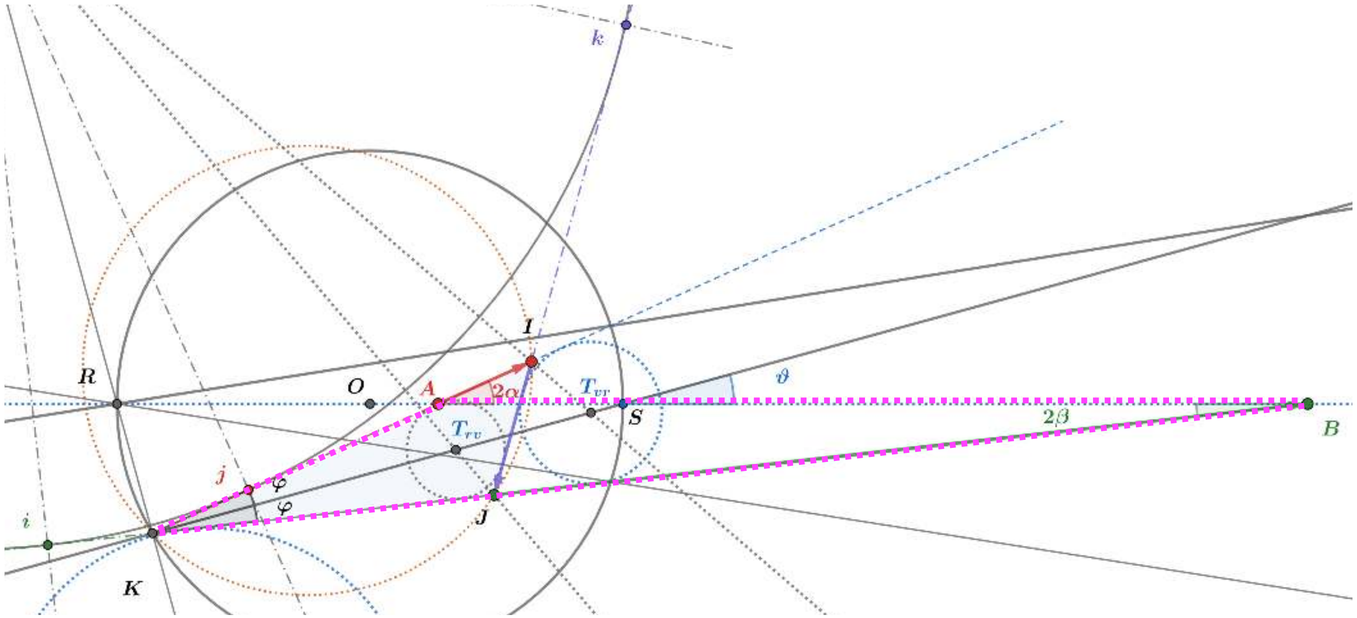
$$\begin{aligned}
2a &= AI + IJ + JB \\
&= AI + (Jk - Ik) + JB \\
&= (AI - jI) + (iJ + JB) \\
&= iB - jA \\
&= KB + iK - (KA - Kj)
\end{aligned}$$

or further

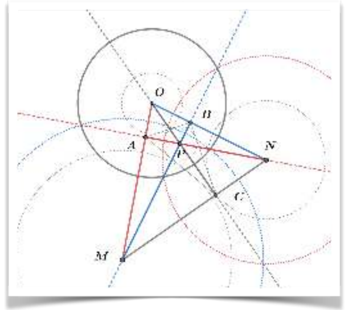
$$KB - KA = 2a - 2K_{rr}$$

with $K_{rr} = iK = Kj$.

This observation is independent of I, J.



Conjugate Mirrors

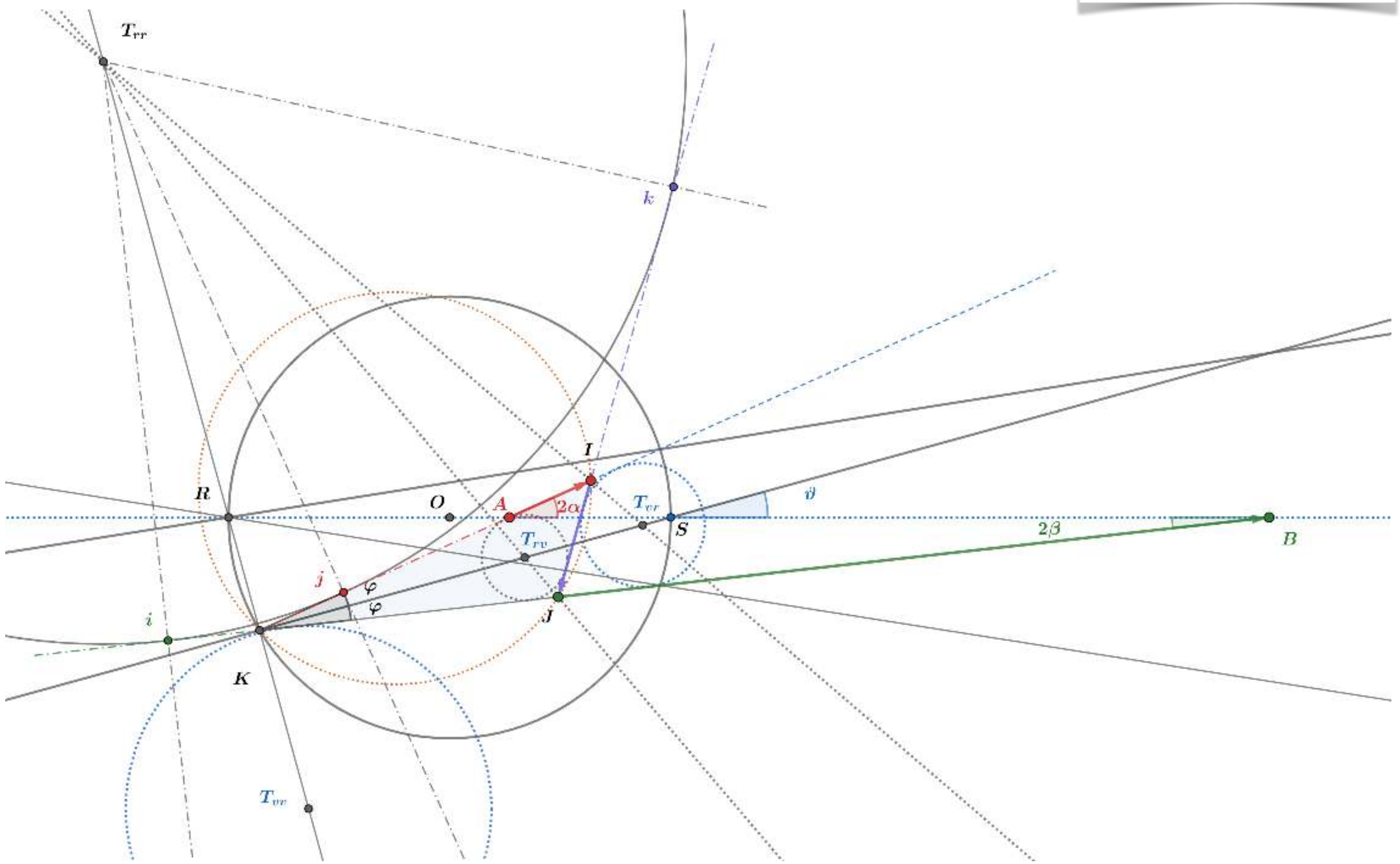


Furthermore we get

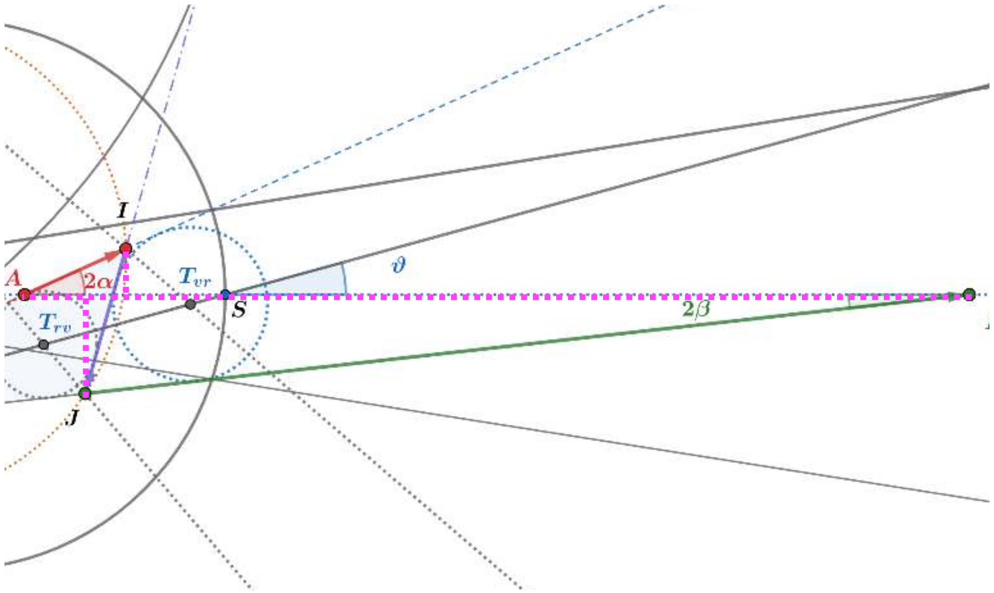
$$\begin{array}{ccccccccccccc} I_{vv} & = & J_{rr} & = & K_{rv} & = & I & + & J & + & K \\ I_{vr} & = & J_{rv} & = & K_{rr} & = & -I & + & J & + & K \\ I_{rv} & = & J_{vr} & = & K_{vv} & = & I & - & J & + & K \\ I_{rr} & = & J_{vv} & = & K_{vr} & = & I & + & J & - & K \end{array}$$

for the distances of I, J, K to the inscribed and escribed circles T_{**} to triangle IJK. From observations such as $K_{rr} + I_{rr} = Kj + jI = \overline{KI}$, we identify

$\overline{KI} = 2I, \quad \overline{IJ} = 2K, \quad \overline{JK} = 2J$



Conjugate Mirrors



For the distance IJ we find

$$\begin{aligned} IJ^2 = w^2 &= (2a-u-v)^2 \\ &= (u\cos 2\alpha + v\cos 2\beta - 2c)^2 + (u\sin 2\alpha + v\sin 2\beta)^2 \end{aligned}$$ and thus

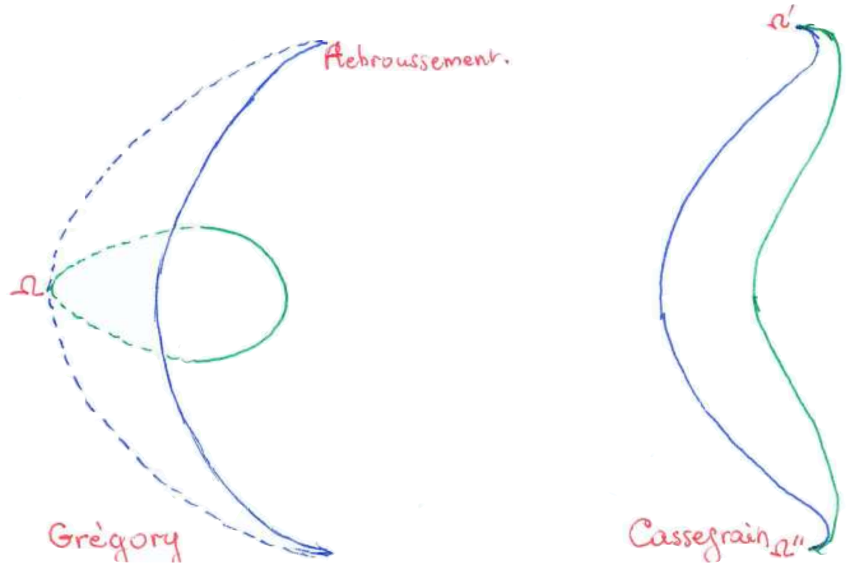
$$\begin{aligned} w^2 &= 4a^2 - 4a(u+v) + (u^2+v^2+2uv) \\ &= u^2+v^2+4c^2+2uv\cos(2\alpha-2\beta) - 4cu\cos 2\alpha - 4cv\cos 2\beta \\ 0 &= 4(a^2-c^2) - 4u(a - c\cos 2\alpha) - 4v(a - c\cos 2\beta) + 2uv(1-\cos(2\alpha-2\beta)) \\ 0 &= (a^2-c^2) - (a - c\cos^2\alpha + c\sin^2\alpha)u - ((a-c)\cos^2\beta + (a+c)\sin^2\beta)v \end{aligned}$$ and

$$\begin{aligned} &+\tfrac{1}{2}(1-(\cos 2\alpha\cos 2\beta + \sin 2\alpha\sin 2\beta))uv \\ &= \frac{1}{uv} - \left[\frac{\cos^2\alpha}{a+c} + \frac{\sin^2\alpha}{a-c}\right]\frac{1}{v} - \left[\frac{\cos^2\beta}{a+c} + \frac{\sin^2\beta}{a-c}\right]\frac{1}{u} + \frac{\sin^2(\alpha-\beta)}{a^2-c^2} \end{aligned}$$

finally another polar form:

$$\left[\frac{1}{u} - \frac{\cos^2\alpha}{a+c} - \frac{\sin^2\alpha}{a-c}\right]\cdot\left[\frac{1}{v} - \frac{\cos^2\beta}{a+c} - \frac{\sin^2\beta}{a-c}\right] = \left[\frac{\cos\alpha\cos\beta}{a+c} \pm \frac{\sin\alpha\sin\beta}{a-c}\right]^2$$

(Gregory telescope with positive sign, Cassegrain with negative sign)



(de Casteljau, letter to Farouki, 1991)

Conjugate Mirrors

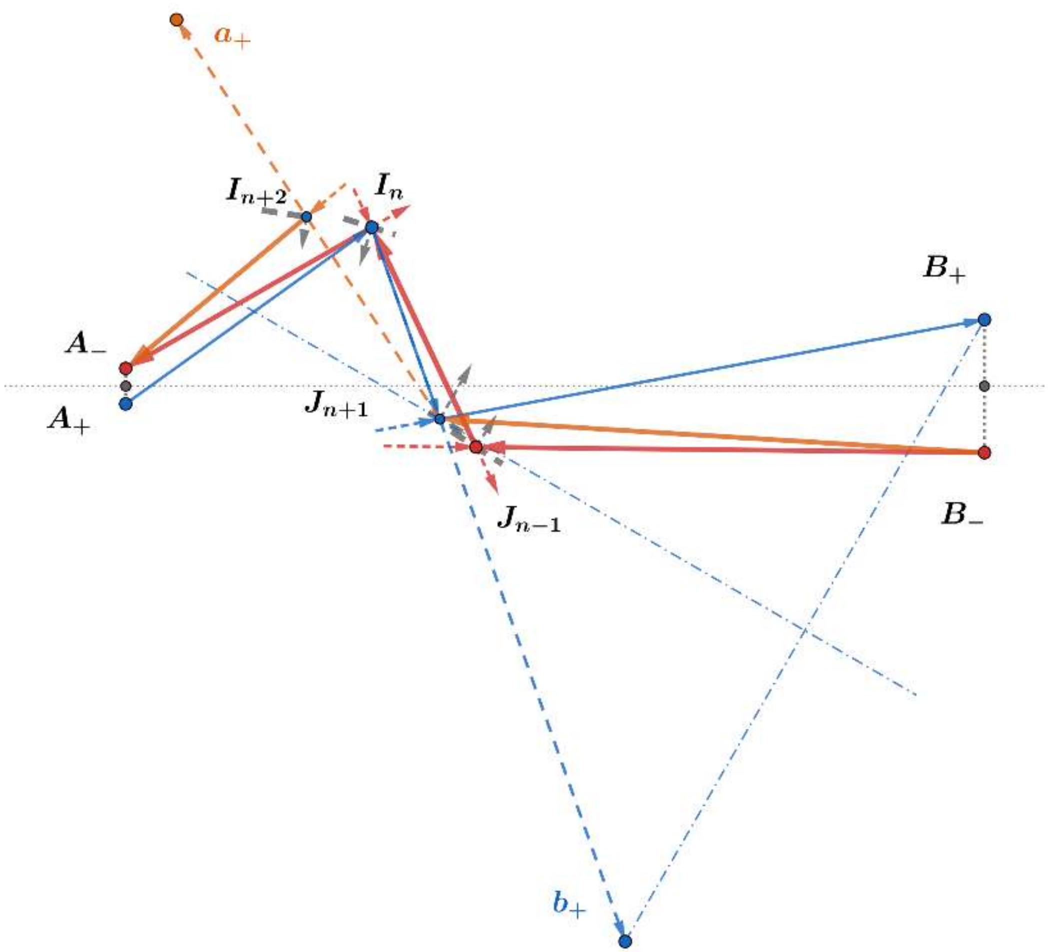
The 'rebound'

Let us construct the unknown conjugate mirror points I_k, J_k:

Given $B\_J_{n-1}I_nA\_$, we can travel back an identical length from A_+ to B_+ (Fermat's principle):

At I_n the ray is reflected along the I_n-tangent to b_+. If reflected at some (unknown) J_{n+1}, the ray returns to B_+, so I_nb_+ is equal to $I_nJ_{n+1}B_+$. From the earlier duality we get the section bisector b_+B_+ is the angle bisector at a reflection. So, the intersection of b_+B_+ with I_nb_+ delivers the position of J_{n+1}.

We repeat the construction with the ray B_J_{n+1} to find I_{n+2} at the intersection of $J_{n+1}a_+$ with the section bisector $a_+A\_$ etc.



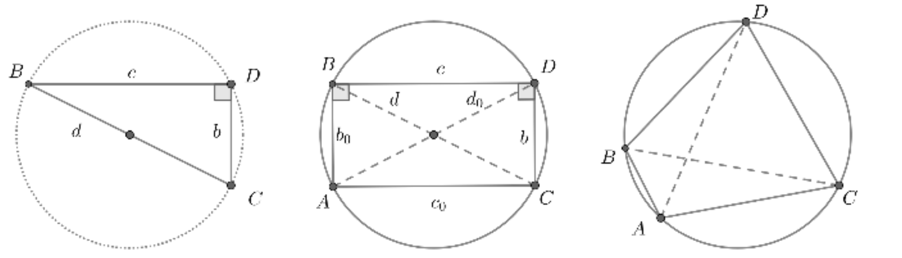
(de Casteljau,, 1991)

Regular Polygons

"The" algorithm
Algebraic smoothing
Metric geometry:
    focal splines, 14-point strophoid, conjugate mirrors, regular polygons
16D+3C

Regular Polygons

Pythagoras, Ptolemy, Polygons



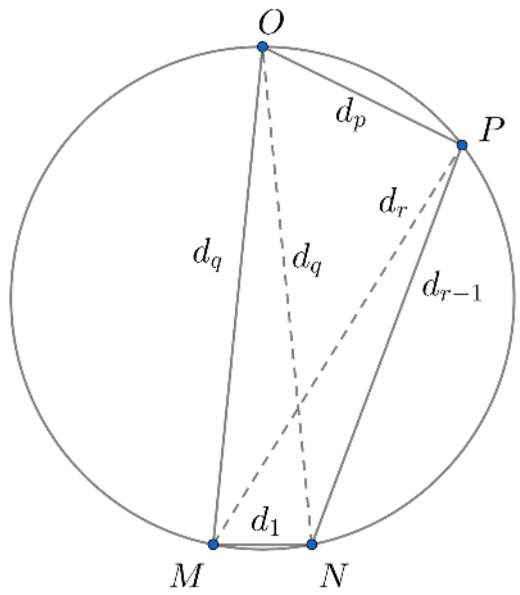
$d^2 = b^2 + c^2$ $\quad$ $dd_0 = bb_0 + cc_0$ $\quad$ $DA.BC = DB.CA + DC.BA$

Ptolemy's theorem is a polar form of Pythagoras' theorem:

$$ON\,.\,MP = OP\,.\,MN + OM\,.\,PN$$

While P walks along (half) the circle / polygon, we get several relations:

$$\begin{array}{rcl} d_q d_r & = & d_p d_1 + d_q d_{r-1} \\ d_q d_{r-1} & = & d_{p+1} d_1 + d_q d_{r-2} \\ \vdots & & \\ d_q d_2 & = & d_{q-1} d_1 + d_q d_1 \end{array} \quad \rightarrow \quad \begin{array}{rcl} d_q(d_r - d_{r-1}) & = & d_p d_1 \\ d_q(d_{r-1} - d_{r-2}) & = & d_{p+1} d_1 \\ & \vdots & \\ d_q(d_2 - d_1) & = & d_{q-1} d_1 \\ \hline d_q(d_r - d_1) & = & d_1 \sum_{q-1}^{p} d_i \\ d_q d_r & = & d_1 \sum_{q}^{p} d_i \end{array}$$

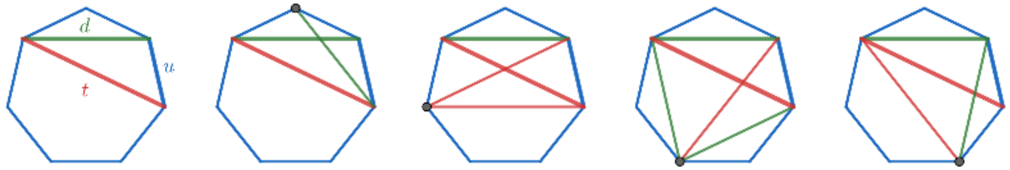


Theorem (golden ratios)
The diagonal lengths d_i in an n-gon, $n = p + q + r$, fulfil the equation

$$\frac{d_r - d_{r-1}}{d_p} = \frac{d_{r-1} - d_{r-2}}{d_{p+1}} = \cdots = \frac{d_2 - d_1}{d_{q-1}} = \frac{d_1}{d_q} = \frac{d_r}{\sum_{i=p}^{q} d_i}$$

Regular Polygons

n=7



$u^2+ut=d^2$ $\quad$ $u^2+dt=t^2$ $\quad$ $ud+d^2=t^2$ $\quad$ $ut+ud=dt$

For a heptagon, we thus get the relations $\frac{t-d}{u}=\frac{d-u}{d}=\frac{u}{t}=\frac{t}{u+d+t}$, where the variables represent the polygon diagonals ordered in french numbers *un*, *deux*, *trois*.

The Ptolemy relations deliver (P walks from 12h down to 5h): $\begin{aligned} d^2 &= u^2+ut & (I)\\ u^2 &= t^2-dt & (II)\\ t^2 &= d^2+ud & (III)\\ dt &= ut+ud & (O)\end{aligned}$ as a matrix: $[u \quad d \quad t]\begin{bmatrix}u\\d\\t\end{bmatrix}=\begin{bmatrix}u^2 & t^2-d^2 & d^2-u^2\\ ud & d^2 & t^2-u^2\\ ut & dt & t^2\end{bmatrix}$

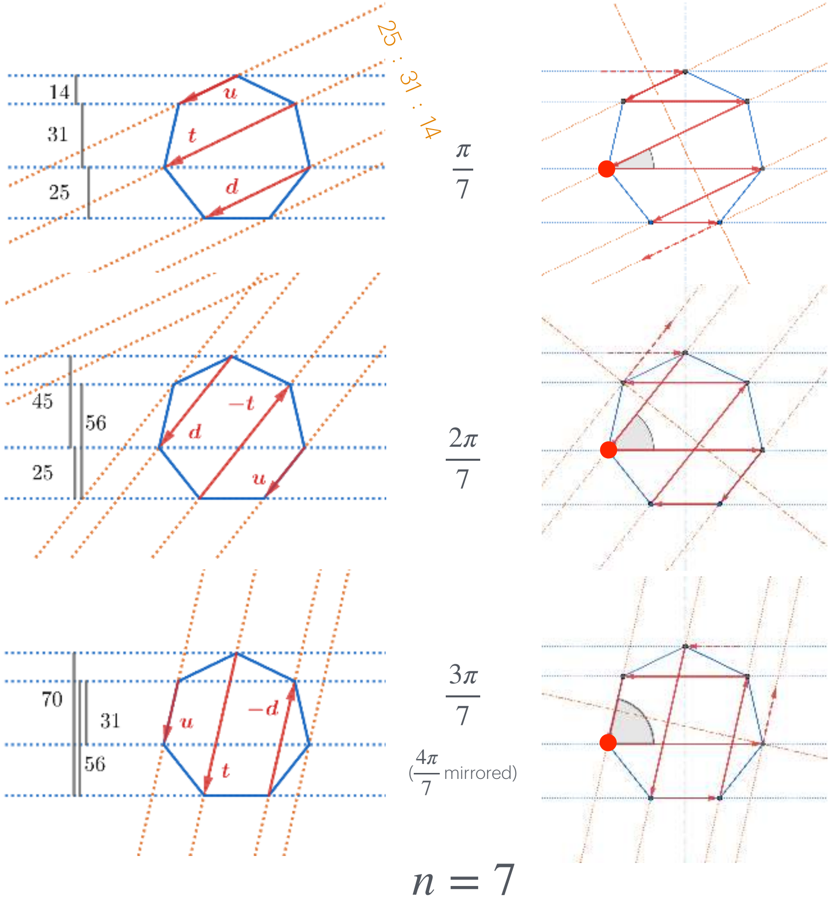
Or $\begin{aligned}\frac{t^2-d^2}{d} &= \frac{dt}{d+t} = u \quad (III)\,\&\,(O)\\ \frac{t^2-u^2}{t} &= \frac{ut}{t-u} = d \quad (II)\,\&\,(O)\\ \frac{d^2-u^2}{u} &= \frac{ud}{d-u} = t \quad (I)\,\&\,(O)\end{aligned}$, which leads to $\begin{aligned}(t+d)(t-d)(t+d)-d^2t &= t^3+t^2d-2d^2t-d^3 &= 0\\ (t-u)(t+u)(t-u)-t^2u &= u^3-u^2t-2t^2u+t^3 &= 0\\ (d-u)(d+u)(d-u)-u^2d &= d^3-d^2u-2u^2d+u^3 &= 0\\ \text{thus} \qquad & x^3+x^2-2x-1 &= 0\end{aligned}$ with solutions $\begin{aligned}x_1 &= \tfrac{t}{d}\\ x_2 &= \tfrac{-u}{t}\\ x_3 &= \tfrac{d}{-u}\end{aligned}$

Due to a recursion originated by de Casteljau, $U=t, D=d+t, T=u+d+t$, we get better approximations $\frac{T}{D},\frac{-U}{T},\frac{D}{-U}$ and thus a sequence of matrices

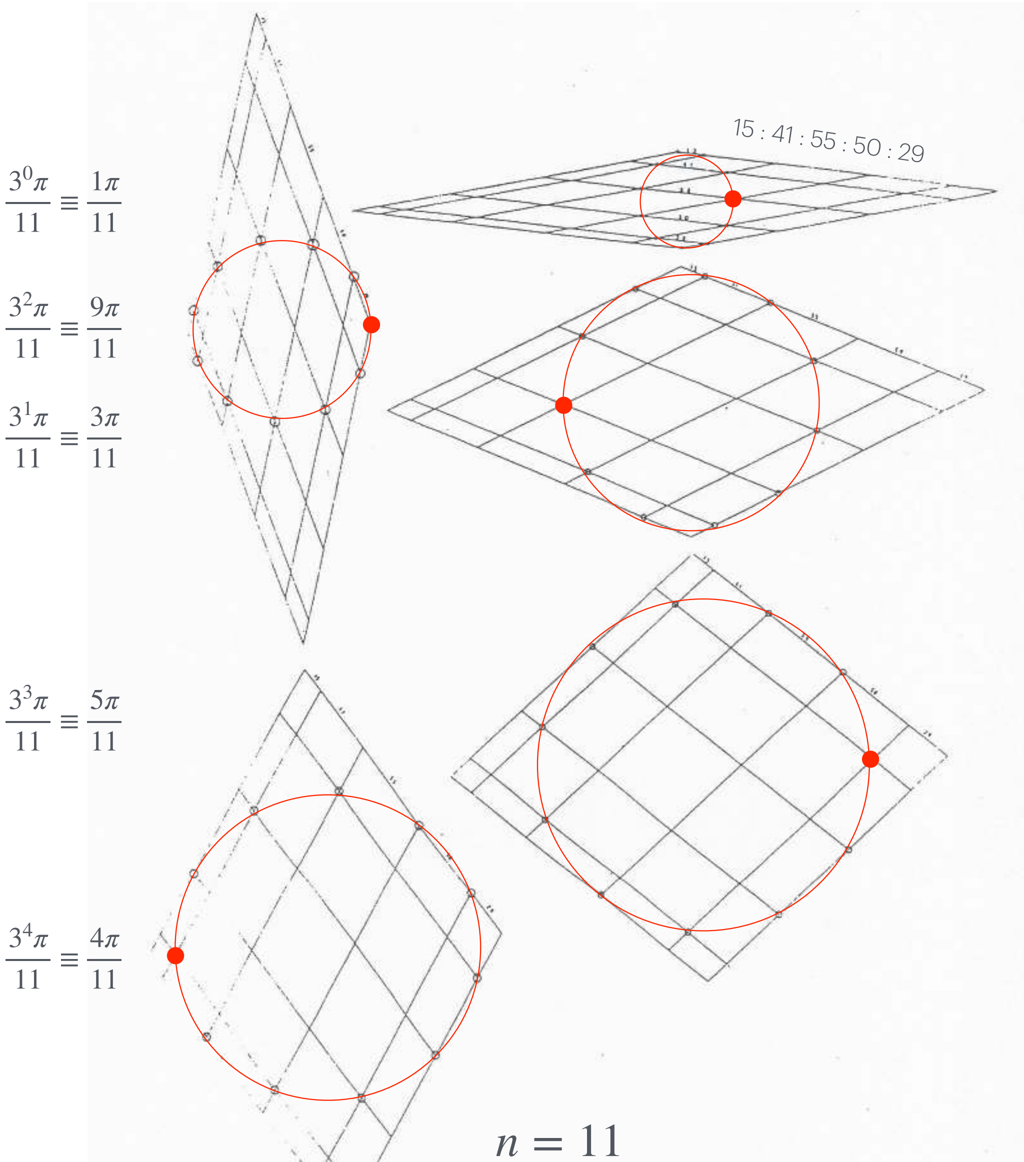
$$M=M^1=\begin{bmatrix}0&0&1\\0&1&1\\1&1&1\end{bmatrix},\quad M^2=\begin{bmatrix}1&1&1\\1&2&2\\1&2&3\end{bmatrix},\quad M^3=\begin{bmatrix}1&2&3\\2&4&5\\3&5&6\end{bmatrix},\quad \dots,\quad M^6=\begin{bmatrix}\mathbf{14}&\mathbf{25}&\mathbf{31}\\ \mathbf{25}&\mathbf{45}&\mathbf{56}\\ \mathbf{31}&\mathbf{56}&\mathbf{70}\end{bmatrix},\quad \dots,\quad M^9=\begin{bmatrix}157&283&353\\283&510&636\\353&636&793\end{bmatrix},\quad \dots,\quad M^\infty=\begin{bmatrix}u^2 & t^2-d^2 & d^2-u^2\\ ud & d^2 & t^2-u^2\\ ut & dt & t^2\end{bmatrix}$$

Regular Polygons

Angles and Meshes



(Müller 2024)

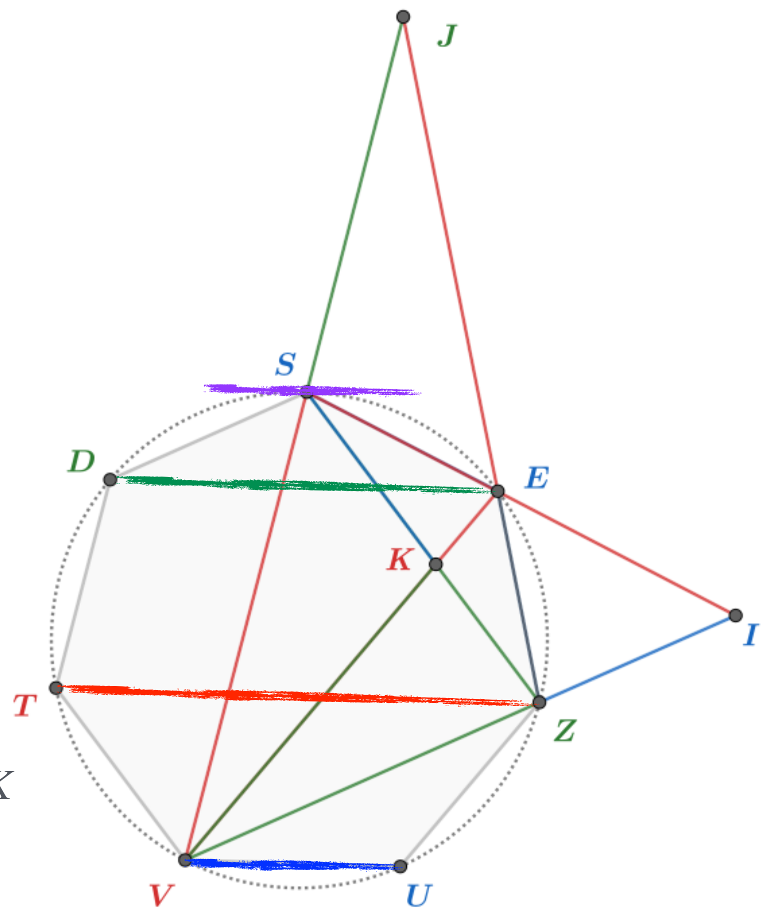


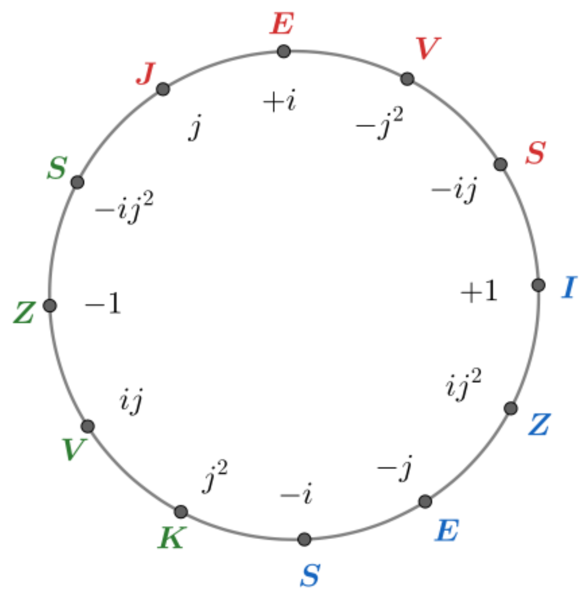
(de Casteljau 2001)

The heptagon's complete quadrilateral

Seven points:
Sommet, **U**n, **D**eux, **T**rois
and **E**ins, **Z**wei, **V**ier

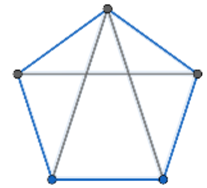
We observe:
- four times a sequence with diagonal lengths u $(IZESK)$, d $(KVZSJ)$, t $(JEVSI)$
- each angle in the clockwise sequence is ij^2-times the preceding one ($i^4 = 1 = j^3$)
- similar triangles JES, SIZ, VEZ, VSK





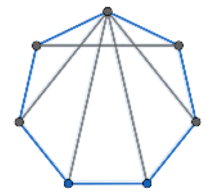
Ratios beyond Golden

Two lengths $a < b$ are in a golden ratio, iff $\frac{a+b}{b} = \frac{b}{a} = \Phi$, or $\begin{bmatrix} b \\ a+b \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} [a \quad b]$.



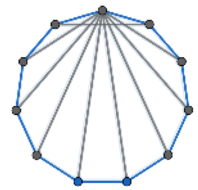
The sequence of matrices $M = M^1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad M^2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad \ldots, \quad M^5 = \begin{bmatrix} 3 & 5 \\ 5 & 8 \end{bmatrix}, \quad \ldots, \quad M^{10} = \begin{bmatrix} 34 & 55 \\ 55 & 89 \end{bmatrix}, \quad \ldots, \quad M^{\infty} = \begin{bmatrix} a^2 & b^2 - a^2 \\ ab & b^2 \end{bmatrix}$

approximates $\Phi = \frac{b^2 - a^2}{b^2 - ab} = \frac{a+b}{b} = \frac{b}{a} = \frac{ab}{a^2}$, where Φ is solution of the quadratic equation $\Phi^2 = \Phi + 1$



Three lengths $u < d < t$ approximate the solutions of $x^3 + x^2 - 2x - 1$: $x_1 = \frac{t}{d} \quad x_2 = \frac{-u}{t} \quad x_3 = \frac{d}{-u}, \quad M^5 = \begin{bmatrix} 6 & 11 & 14 \\ 11 & 20 & 25 \\ 14 & 25 & 31 \end{bmatrix}$

Five lengths $u < d < t < q < c$ approximate the roots of $x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1 = 0$: $x_1 = \frac{c}{u} \quad x_2 = \frac{-q}{t} \quad x_3 = \frac{u}{d} \quad x_4 = \frac{t}{c} \quad x_5 = \frac{d}{-q}, \quad M^4 = \begin{bmatrix} 5 & 9 & 12 & 14 & 15 \\ 9 & 17 & 23 & 27 & 29 \\ 12 & 23 & 32 & 38 & 41 \\ 14 & 27 & 38 & 46 & 50 \\ 15 & 29 & 41 & 50 & 55 \end{bmatrix}$



16D+3C

“The” algorithm
Algebraic smoothing
Metric geometry:
    focal splines, 14-point strophoid, conjugate mirrors, regular polygons
16D+3C

16D+3C *(16 droites + 3 coniques)*

"A picture is worth a thousand words" - yet it's about combinatorics

Desargues (1636)

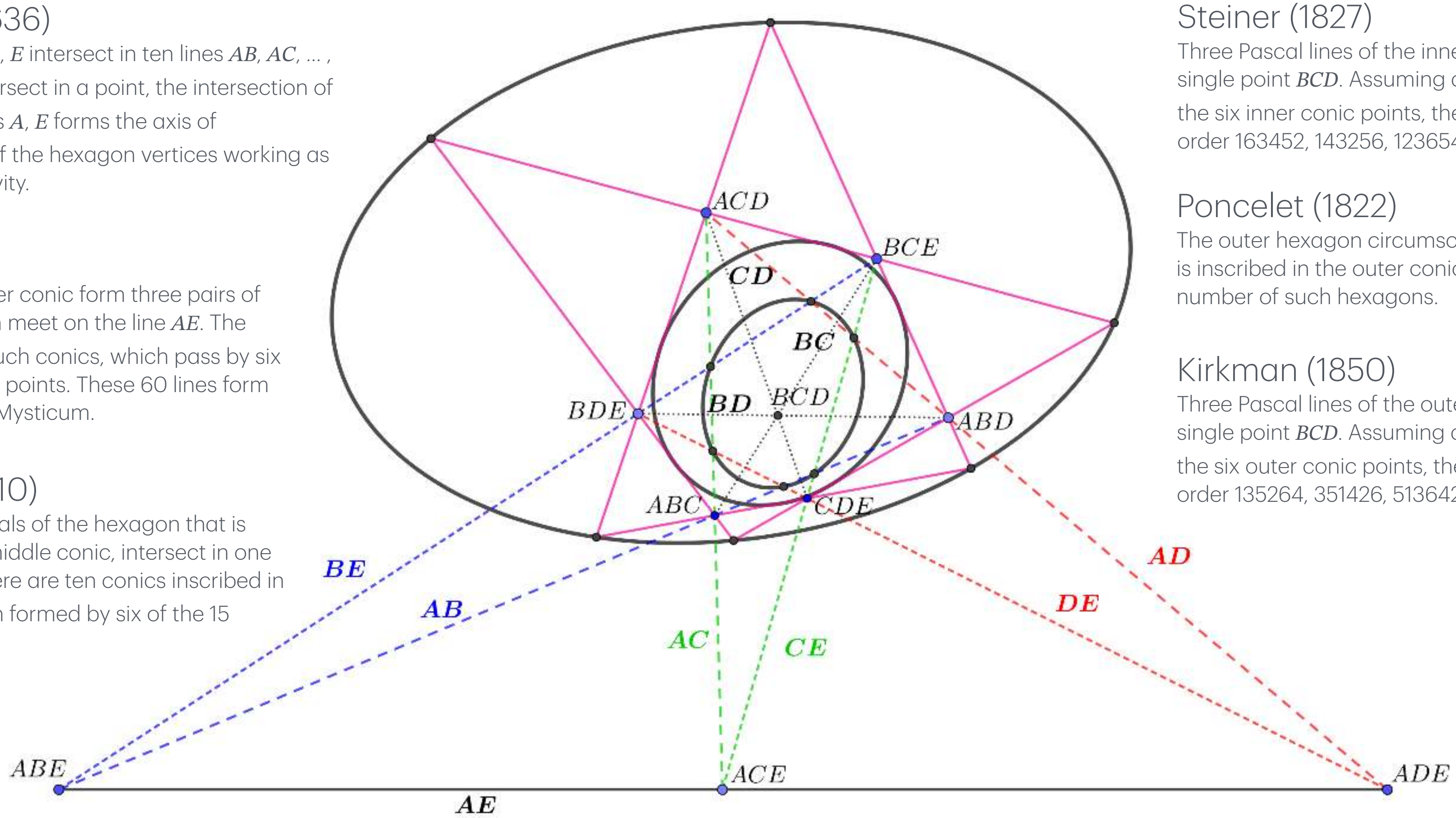
Five planes A, B, C, D, E intersect in ten lines AB, AC, ... , DE. While B, C, D intersect in a point, the intersection of the remaining planes A, E forms the axis of perspectivity, each of the hexagon vertices working as centres of perspectivity.

Pascal (1639)

Six points on the inner conic form three pairs of opposite sides which meet on the line AE. The figure includes ten such conics, which pass by six of the 15 intersection points. These 60 lines form the Hexagrammum Mysticum.

Brianchon (1810)

The principal diagonals of the hexagon that is circumscribing the middle conic, intersect in one single point BCD. There are ten conics inscribed in such hexagons, each formed by six of the 15 possible lines.



Steiner (1827)

Three Pascal lines of the inner conic intersect at one single point BCD. Assuming a clockwise numbering of the six inner conic points, these hexagons are in the order 163452, 143256, 123654.

Poncelet (1822)

The outer hexagon circumscribes the middle conic and is inscribed in the outer conic. Thus, there is an infinite number of such hexagons.

Kirkman (1850)

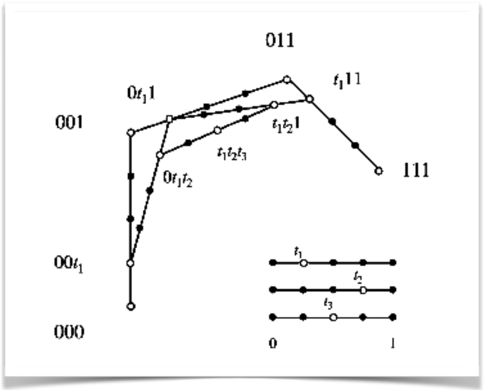
Three Pascal lines of the outer conic intersect at one single point BCD. Assuming a clockwise numbering of the six outer conic points, these hexagons are in the order 135264, 351426, 513642.

(de Casteljau 1999)

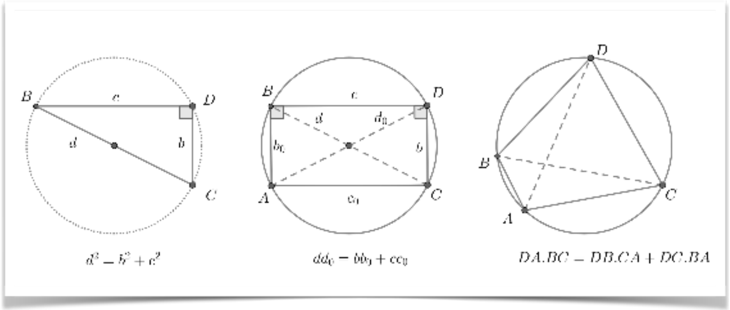
Review

Whilst Paul de Casteljau is now famous for his fundamental algorithm for the generation of curves and surfaces, little is known about his other findings. This talk provides insight into some of his lesser known geometric discoveries. In connection with geometry, his classical algorithm is reviewed as an **index reduction of a polar form**. This idea is used to show de Casteljau's **algebraic** way of **smoothing**, which long went unnoticed. The talk will also unveil unpublished material on metric geometry, covering theoretical advances such as the **14-point strophoid**. Practical applications, including a recurrence approach for **conjugate mirrors** in geometric optics, will also be discussed. A view on **regular polygons** leads to an approximation of their diagonals using **golden matrices**, a generalisation of the golden ratio ... and of Gauss' 17-sided polygon to any regular n-gon.

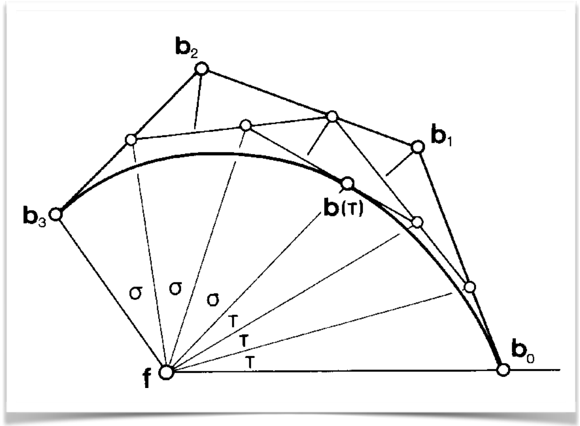
Review: The many polar forms



polynomial polar form



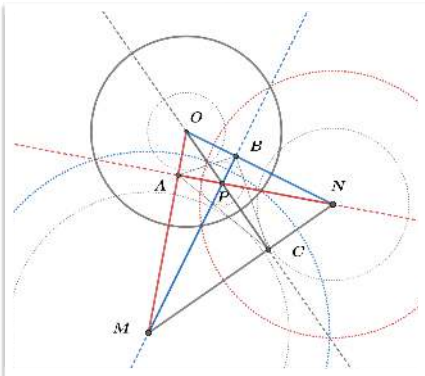
Ptolemy as polar form of Pythagoras



focal polar form

$$
\begin{array}{llllll}
RR: & \left[\frac{1}{u}-\frac{\sin^2\alpha}{a+c}-\frac{\cos^2\alpha}{a-c}\right] & \cdot & \left[\frac{1}{v}-\frac{\sin^2\beta}{a+c}-\frac{\cos^2\beta}{a-c}\right] & = & \left[\pm\frac{\sin\alpha\sin\beta}{a+c}+\frac{\cos\alpha\cos\beta}{a-c}\right]^2 \\
VR: & \left[\frac{1}{u}+\frac{\cos^2\alpha}{a+c}+\frac{\sin^2\alpha}{a-c}\right] & \cdot & \left[\frac{1}{v}-\frac{\sin^2\beta}{a+c}-\frac{\cos^2\beta}{a-c}\right] & + & \left[\pm\frac{\cos\alpha\sin\beta}{a+c}-\frac{\sin\alpha\cos\beta}{a-c}\right]^2 = 0 \\
VV: & \left[\frac{1}{u}+\frac{\cos^2\alpha}{a+c}+\frac{\sin^2\alpha}{a-c}\right] & \cdot & \left[\frac{1}{v}+\frac{\cos^2\beta}{a+c}+\frac{\sin^2\beta}{a-c}\right] & = & \left[\frac{\cos\alpha\cos\beta}{a+c}\pm\frac{\sin\alpha\sin\beta}{a-c}\right]^2 \\
RV: & \left[\frac{1}{u}-\frac{\sin^2\alpha}{a+c}-\frac{\cos^2\alpha}{a-c}\right] & \cdot & \left[\frac{1}{v}+\frac{\cos^2\beta}{a+c}+\frac{\sin^2\beta}{a-c}\right] & + & \left[\frac{\sin\alpha\cos\beta}{a+c}\mp\frac{\cos\alpha\sin\beta}{a-c}\right]^2 = 0
\end{array}
$$

(bi-)polar form in conjugate mirrors



heights as metric polar form
lead to orthocentric tetragram

$$
\begin{array}{llll}
i=1: & a_0 = a_1 x & \rightarrow\ x_1 = \dfrac{a_0}{a_1} & \text{(Newton)} \\
i=2: & a_0 = (a_1 + a_2 x_1)x & \rightarrow\ x_2 = \dfrac{a_0}{a_1 + a_2 \dfrac{a_0}{a_1}} & \text{(Whittaker)} \\
i=3: & a_0 = (a_1 + (a_2 + a_3 x_1) x_2) x & \rightarrow\ x_3 = \dfrac{a_0}{a_1 + a_2 \dfrac{a_0}{a_2 + a_3 \dfrac{a_1}{a_2}}} & \text{(continued fractions)}
\end{array}
$$

polar form in analysis

My greatest satisfaction will be to see this presentation inspire new works, complements, or applications, regardless of their field of application.

de Casteljau (1986)

Paul de Casteljau after his presentation of Focal Splines in Chamonix, 1993 (right: Jean Charles Fiorot)

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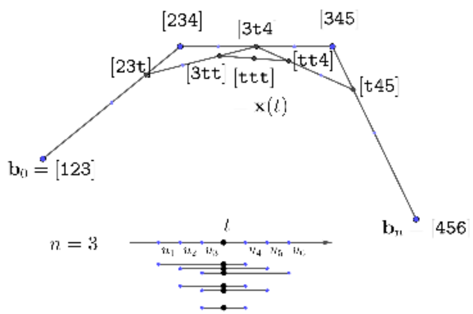
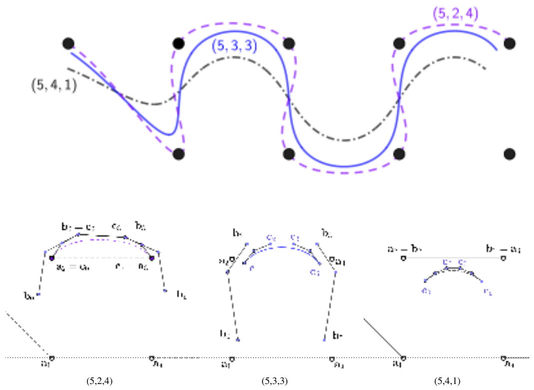
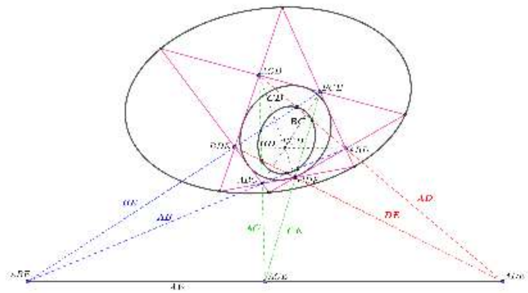


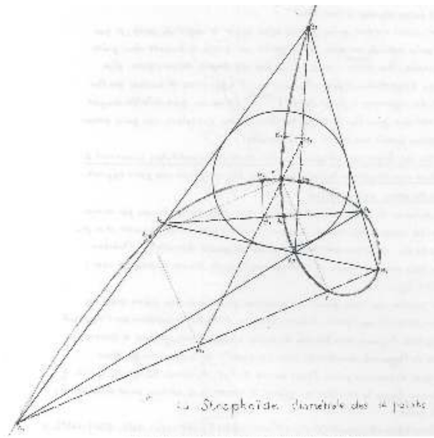
«Formes à Pôles»



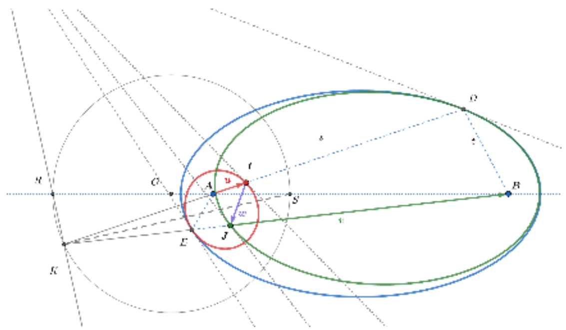
Algebraic Smoothing



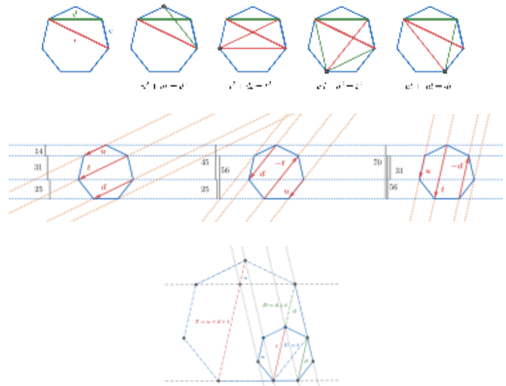
16 lines & 3 conics



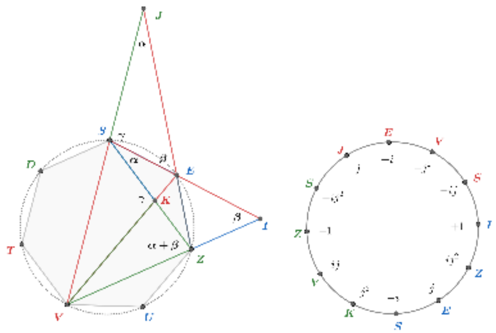
14-point strophoid



Geometric Optics



Beyond the Golden Ratio



Heptagon

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